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Supplement of

Evaluating the enhanced sampling rate for turbulence measurement with a wind lidar profiler

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S1 Problem settings

We consider a resolution cell of the lidar. The origin of the coordinate frame is located at the center of the resolution cell. The mean wind $\mathbf{U}$ is blowing in the x direction, with strength U. The beam pointing vector is $\mathbf{b}$, and assumed uniform over the sampling volume (the beam divergence is neglected). The weighting in the along-beam direction is Gaussian with width σ_l, and Gaussian in the cross-beam direction with width σ_r. The instrument averages for Δt seconds every T_b seconds. The instantaneous wind velocity is $\mathbf{v}(\mathbf{x}, t)$. According to the frozen-turbulence hypothesis, $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x} - \mathbf{U}t)$.
Altogether, output sample number n can be expressed as:

$$v_n = \frac{1}{\Delta t} \iiint \int_{nT_b - \Delta t/2}^{nT_b + \Delta t/2} \mathbf{b} \cdot \mathbf{v}(\mathbf{x}, t) W(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \tag{S1}$$

$$= \frac{\mathbf{b}}{\Delta t (2\pi)^{3/2} \sigma_l \sigma_r^2} \cdot \iiint \int_{nT_b - \Delta t/2}^{nT_b + \Delta t/2} \mathbf{v}(\mathbf{x}, t) e^{-\frac{(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_l^2} - \frac{\|\mathbf{x}\|^2 - (\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_r^2}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \tag{S2}$$

Our aim is to compute the transfer function of the filtering effect induced by the instrument acquisition timing and geometry. What is the size of the flow structures that are adequately sampled by the instrument? In this contribution we consider only the filtering effect of averaging over the sampling volume and interval. A more in-depth study definitely should consider the effect of measurement noise caused by physical effects such as detector noise, speckle noise, or backscattering material concentration variations within the sampling volume. Consideration of these effects would require a lot of information from the instrument manufacturer and is left for future work.

We introduce the spatial Fourier transform of the wind velocity

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \iiint e^{i\mathbf{k}\cdot(\mathbf{x} - \mathbf{U}t)} \hat{\mathbf{v}}(\mathbf{k}) \mathrm{d}\mathbf{k}. \tag{S3}$$

An ideal instrument providing perfectly pointwise and instantaneous measurements of the wind velocity at the sampling volume center would provide:

$$v_n^{ideal} = \frac{\mathbf{b}}{(2\pi)^3} \cdot \iiint \hat{\mathbf{v}}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{U}nT_b} \mathrm{d}\mathbf{k}. \tag{S4}$$

Due to the finite sampling volume size and finite observation interval, however, the real instrument provides

$$v_n = \frac{\mathbf{b}}{\Delta t (2\pi)^{9/2} \sigma_l \sigma_r^2} \cdot \iiint \hat{\mathbf{v}}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{U}nT_b} \iiint \int_{-\Delta t/2}^{\Delta t/2} e^{i\mathbf{k}\cdot(\mathbf{x} - \mathbf{U}t)} e^{-\frac{(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_l^2} - \frac{\|\mathbf{x}\|^2 - (\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_r^2}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \, \mathrm{d}\mathbf{k}. \tag{S5}$$

This can be seen as the result of the action on the true velocity field of a spectral-domain filtering operator

$$v_n = \frac{\mathbf{b}}{(2\pi)^3} \cdot \iiint \hat{\mathbf{v}}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{U}nT_b} \, H(\mathbf{k}) \, \mathrm{d}\mathbf{k}, \tag{S6}$$

with transfer function

$$H(\mathbf{k}) = \frac{1}{\Delta t (2\pi)^{3/2} \sigma_l \sigma_r^2} \iiint \int_{-\Delta t/2}^{\Delta t/2} e^{i\mathbf{k}\cdot(\mathbf{x} - \mathbf{U}t)} e^{-\frac{(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_l^2} - \frac{\|\mathbf{x}\|^2 - (\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_r^2}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \tag{S7}$$

S2 Evaluation of the transfer function

The temporal windowing effect is easily computed.

$$H(\mathbf{k}) = \frac{1}{\Delta t(2\pi)^{3/2}\sigma_l\sigma_r^2}\iiint e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\frac{(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_l^2}-\frac{\|\mathbf{x}\|^2-(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_r^2}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{k}\int_{-\Delta t/2}^{\Delta t/2}e^{-i\mathbf{k}\cdot\mathbf{U}t}\mathrm{d}t \tag{S8}$$

$$= \frac{1}{\Delta t(2\pi)^{3/2}\sigma_l\sigma_r^2}\iiint e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\frac{(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_l^2}-\frac{\|\mathbf{x}\|^2-(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_r^2}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{k}\left[\frac{e^{i\mathbf{k}\cdot\mathbf{U}t}}{i\mathbf{k}\cdot\mathbf{U}}\right]_{-\Delta t/2}^{\Delta t/2} \tag{S9}$$

$$= \frac{1}{(2\pi)^{3/2}\sigma_l\sigma_r^2}\mathrm{sinc}\left(\frac{\Delta t}{2}\mathbf{k}\cdot\mathbf{U}\right)\iiint e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\frac{(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_l^2}-\frac{\|\mathbf{x}\|^2-(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_r^2}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{k} \tag{S10}$$

The spatial filtering effect requires the computation of a Gaussian integral over 3-dimensional space. The eigenvectors and eigenvalues of the matrix characterizing the weighting pattern, $\mathbf{M}$, are known to be $\mathbf{b}$, with eigenvalue σ_l^2, and any pair of vectors orthogonal to $\mathbf{b}$, with eigenvalues σ_r^2. The computation then proceeds as follows:

$$W(\mathbf{k}) = \iiint e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\frac{(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_l^2}-\frac{\|\mathbf{x}\|^2-(\mathbf{x}\cdot\mathbf{b})^2}{2\sigma_r^2}}\,\mathrm{d}\mathbf{x} \tag{S11}$$

$$= \iiint e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\frac{\mathbf{x}^T\mathbf{M}\mathbf{x}}{2}}\,\mathrm{d}\mathbf{x} \tag{S12}$$

$$= \iiint e^{-\frac{\mathbf{x}^T\mathbf{M}\mathbf{x}-2i\mathbf{k}\cdot\mathbf{x}}{2}}\,\mathrm{d}\mathbf{x} \tag{S13}$$

$$= \iiint e^{-\frac{(\mathbf{x}-i\mathbf{M}^{-1}\mathbf{k})^T\mathbf{M}(\mathbf{x}-i\mathbf{M}^{-1}\mathbf{k})+\mathbf{k}^T\mathbf{M}^{-1}\mathbf{k}}{2}}\,\mathrm{d}\mathbf{x} \tag{S14}$$

$$= e^{-\frac{\mathbf{k}^T\mathbf{M}^{-1}\mathbf{k}}{2}}\iiint e^{-\frac{(\mathbf{x}-i\mathbf{M}^{-1}\mathbf{k})^T\mathbf{M}(\mathbf{x}-i\mathbf{M}^{-1}\mathbf{k})}{2}}\,\mathrm{d}\mathbf{x} \tag{S15}$$

$$= e^{-\frac{\mathbf{k}^T\mathbf{M}^{-1}\mathbf{k}}{2}}\iiint e^{-\frac{\mathbf{x}^T\mathbf{M}\mathbf{x}}{2}}\,\mathrm{d}\mathbf{x} \tag{S16}$$

$$= (2\pi)^{3/2}\sqrt{|\det(\mathbf{M})|}e^{-\frac{\mathbf{k}^T\mathbf{M}^{-1}\mathbf{k}}{2}}. \tag{S17}$$

The final result is obtained as

$$H(\mathbf{k}) = \mathrm{sinc}\left(\frac{\Delta t}{2}\mathbf{k}\cdot\mathbf{U}\right)e^{-\frac{\sigma_l^2(\mathbf{k}\cdot\mathbf{b})^2+\sigma_r^2(\|\mathbf{k}\|^2-(\mathbf{k}\cdot\mathbf{b})^2)}{2}}, \tag{S18}$$

and the squared modulus of the transfer function is

$$|H|^2(\mathbf{k}) = \mathrm{sinc}^2\left(\frac{\Delta t}{2}\mathbf{k}\cdot\mathbf{U}\right)e^{-\left[\sigma_l^2(\mathbf{k}\cdot\mathbf{b})^2+\sigma_r^2(\|\mathbf{k}\|^2-(\mathbf{k}\cdot\mathbf{b})^2)\right]} \tag{S19}$$

S3 Discussion

A number of comments are in order at this point:

- The measurement value is the result of a space-time filtering of the wind velocity field projection along the measurement direction.
- The transfer function includes a part due to time-averaging (the sinc^2 term) and a part due to space-averaging (the Gaussian kernel).

- The time-filtering effect is anisotropic, and the "master direction" is the direction pointed by $\mathbf{U}$. The structures in the wind field that have a wavelength smaller than $\pi \Delta t U$ in the along-wind direction are filtered out.

- The space-filtering effect due to the weighting by the beam is also anisotropic. There the "master direction" is set by the beam pointing vector. Inspection of the Gaussian kernel reveals that structures in the wind field that have a wavelength smaller than σ_l in the along-beam direction are attenuated, and that so are structures that have a wavelength smaller than σ_r in the cross-beam direction. But such structures would be so small that this filtering effect is in fact inconsequential.

- Overall, the effective transfer function is thus:

$$|H|^2(\mathbf{k}) = \mathrm{sinc}^2\left(\frac{\Delta t}{2}\mathbf{k}\cdot\mathbf{U}\right) e^{-\sigma_l^2(\mathbf{k}\cdot\mathbf{b})^2}. \tag{S20}$$

- Finally, the wavevector domain that is unaffected by the filter is the intersection of a slice perpendicular to the direction of $\mathbf{U}$, keeping the structures that are longer than $\pi \Delta t U$, and a slice perpendicular to the direction of $\mathbf{b}$, with all the structures that are longer than σ_l.

\- Plots of the different factors are shown below in Fig. S1.

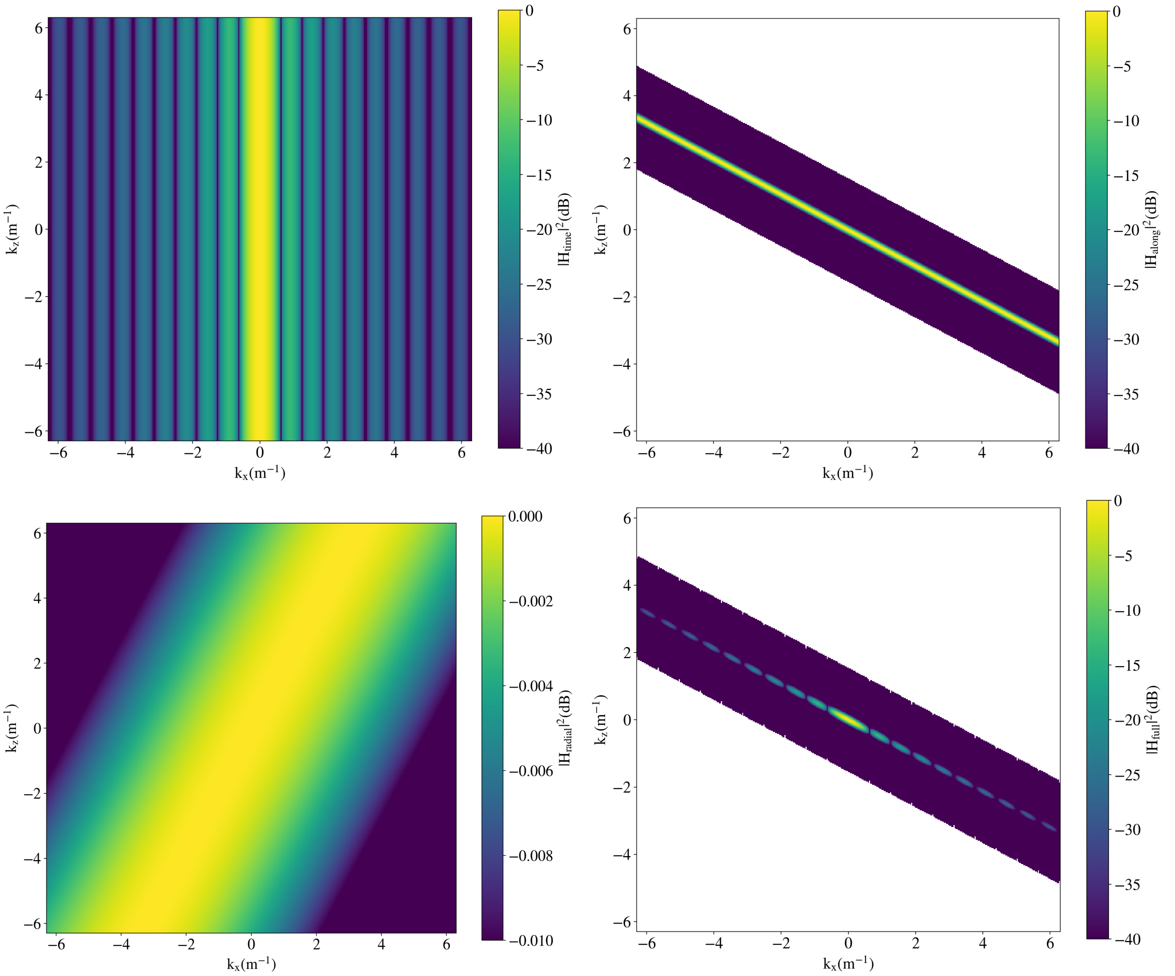

Figure S1: Transfer function factors in the case of a LIDAR averaging during 1 s, pointing at 28° angle from zenith in the plane of a $10\,\mathrm{m}\cdot\mathrm{s}^{-1}$ wind, with a bin length of $20\,\mathrm{m}$ and a beam radius of $1\,\mathrm{cm}$. Top row, left: transfer function contribution associated to the time-averaging. top row, right: transfer function factor associated to the along-beam filtering effect. Bottom row, left: transfer function contribution associated to the cross-beam filtering effect (see very different color scale). Bottom row, right: complete transfer function.