Figure 1. Schematic of three-dimensional velocity lattices. Coordinate-normal planes marked in yellow. Each vector referring to a discrete velocity \( e_{ijk} \) as given in Eq. (1). Velocities of the D3Q19 lattice (Qian et al., 1992) with 19 discrete directions given by orange vectors. Additional velocity directions considered in the D3Q27 lattice given by red vectors.

where

\[
e_{ijk} = (ic, jc, kc)
\]

(2)
is the particle velocity vector and \( i, j, k \in \{-1, 0, 1\} \). The lattice speed \( c \) is chosen such that

\[
c = \Delta x / \Delta t
\]

(3)

On uniform Cartesian grids PDFs are therefore inherently advected from their source (black dot in Fig. 1) to the neighboring nodes during one time step avoiding any interpolation in the advection. The collision operator \( \Omega_{ijk} \) on the right-hand side models the redistribution of \( f \) through particle collisions within the control volume. Based on kinetic theory the collision process is modelled as a relaxation of particle distribution functions towards an equilibrium. In the classical and most simple collision model, the single-relaxation-time model (SRT), commonly referred to as lattice Bhatnagar-Gross-Kroog (LBGK) model (Bhatnagar et al., 1954), all PDFs are relaxed towards an equilibrium using a single constant relaxation time \( \tau \), viz.

\[
\Omega_{ijk}(t, \mathbf{x}) = -\frac{\Delta t}{\tau} \left( f_{ijk}(t, \mathbf{x}) - f_{ijk}^\text{eq}(t, \mathbf{x}) \right)
\]

(4)

The equilibrium distribution \( f_{ijk}^\text{eq} \) is given by the second-order Taylor expansion of the Maxwellian equilibrium

\[
f_{ijk}^\text{eq} = w_{ijk} \rho \left( 1 + \frac{\mathbf{u} \cdot e_{ijk}}{c_s^2} + \frac{(\mathbf{u} \cdot e_{ijk})^2}{2 c_s^4} - \frac{\mathbf{u} \cdot \mathbf{u}}{2 c_s^2} \right)
\]

(5)

where \( c_s \) is the lattice speed of sound and \( \mathbf{u} \) and \( \rho \) the macroscopic velocity and density, respectively. The weights \( w_{ijk} \) are specific to the velocity lattice and ensure mass and momentum conservation of the equilibrium.

Macroscopic quantities can generally be obtained from the raw velocity moments of the PDFs

\[
m_{\alpha\beta\gamma} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} \sum_{k=-1}^{1} (ic)^\alpha (jc)^\beta (kc)^\gamma f_{ijk}
\]

(6)