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Supplement of

Top-level rotor optimisations based on actuator disc theory

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Optimum induction distribution for fixed bending moment

Reference values:

$\rho := 1.225$ $U_0 := 10$ $R_0 := 50$ $a := \frac{1}{3}$ $B := 3$ $\lambda := 8$

$$F(x) := \frac{2}{\pi}\cdot\text{acos}\left[e^{-\frac{(1-x)\cdot B\cdot\lambda}{2\cdot(1-a)}}\right]$$

Prandtl tip loss factor - to switch off make B or λ large

Reference case 1: Standard optimisation for maximum power

$$Cp := \int_0^1 8\cdot a\cdot(1-a)^2\cdot x\cdot F(x)\,dx = 0.549$$

$$Ct := \int_0^1 8\cdot a\cdot(1-a)\cdot x\cdot F(x)\,dx = 0.824$$

$$Cm_r := \int_0^1 8\cdot a\cdot(1-a)\cdot x^2\cdot F(x)\,dx = 0.531$$

$M_0 := 0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot R_0^{\,3}\cdot Cm_r = 1.277\times10^7$ Reference total bending moment

$P_0 := 0.5\cdot\rho\cdot U_0^{\,3}\cdot\pi\cdot R_0^{\,2}\cdot Cp = 2.643\times10^6$ Reference rotor power

$T_0 := 0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot R_0^{\,2}\cdot Ct = 3.964\times10^5$ Reference rotor thrust

Reference case 2: Optimum axial induction to maximise power for variable radius and fixed total moment :

$a := 0.2$ $Ct(a) := 4\cdot a\cdot(1-a)$ $Ct(a) = 0.640$ $Cp(a) := \int_0^1 8\cdot a\cdot(1-a)^2\cdot x\cdot F(x)\,dx$

$Cp(a) = 0.475$

$Cm(a) := \int_0^1 8\cdot a\cdot(1-a)\cdot x^2\cdot F(x)\,dx$ $Cm(a) = 0.382$

$$R_{opt} := \left[\frac{M_0}{\left(0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot Cm(a)\right)}\right]^{\frac{1}{3}} = 55.786$$

$M_0 = 1.277\times10^7$

$$\frac{R_{opt}}{R_0} = 1.116$$

11% increase in radius

$$P := 0.5\cdot\rho\cdot U_0{}^3\cdot\pi\cdot R_{opt}{}^2\cdot Cp(a) = 2.842235 \times 10^6$$

$$\frac{P}{P_0} = 1.076$$

7.6% increase in power

Exactly same results with tip loss on or off as F cancels in M_0 and Cm

Axial induction function:

$N := 10$ $a := \frac{1}{3}$ $p := 1$

Arbitrary starting values for the optimisation except that the solutions for maximisation with small and large N initial values appear to be different.

$$a1(x,N,p) := a\cdot\left(1 - x^N\right)^p$$

A definition of axial induction with 3 DOF (a,N,p) that will tend to zero at x=1 and be positive over [0,1].

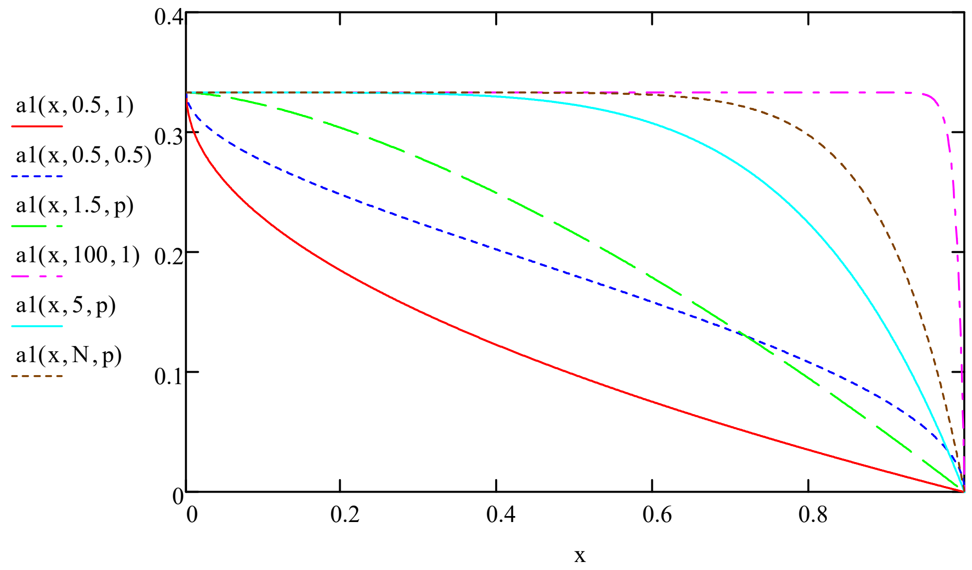


Fig 1 Illustration of variety of possible axial induction distributions

$$P(N,a,p) := \frac{4\cdot\rho\cdot\pi\cdot U_0{}^3\cdot M_0{}^{\frac{2}{3}}\cdot\int_0^1 a\cdot\left(1-x^N\right)^p\cdot\left[1-a\cdot\left(1-x^N\right)^p\right]^2\cdot x\cdot F(x)\,dx}{\left[4\cdot\rho\cdot U_0{}^2\cdot\pi\cdot\int_0^1 a\cdot\left(1-x^N\right)^p\cdot\left[1-a\cdot\left(1-x^N\right)^p\right]\cdot x^2\cdot F(x)\,dx\right]^{\frac{2}{3}}}$$

$$R_{opt}(N,a,p) := \left[\frac{M_0}{\left[4\cdot\rho\cdot U_0{}^2\cdot\pi\cdot\int_0^1 a\cdot\left(1-x^N\right)^p\cdot\left[1-a\cdot\left(1-x^N\right)^p\right]\cdot x^2\cdot F(x)\,dx\right]}\right]^{\frac{1}{3}}$$

Given

$a \le \frac{1}{3}$

$\mathrm{OPT} := \mathrm{Maximize}(P,N,a,p)$ $\qquad \mathrm{OPT} = \begin{pmatrix} 1.504 \\ 0.331 \\ 1.125 \end{pmatrix}$

$a := \mathrm{OPT}_1$ $\qquad a = 0.331$

$N := \mathrm{OPT}_0$ $\qquad N = 1.504$

$p := \mathrm{OPT}_2$ $\qquad p = 1.125$

$$a_{opt}(x,N,a,p) := a\cdot\left(1-x^N\right)^p$$

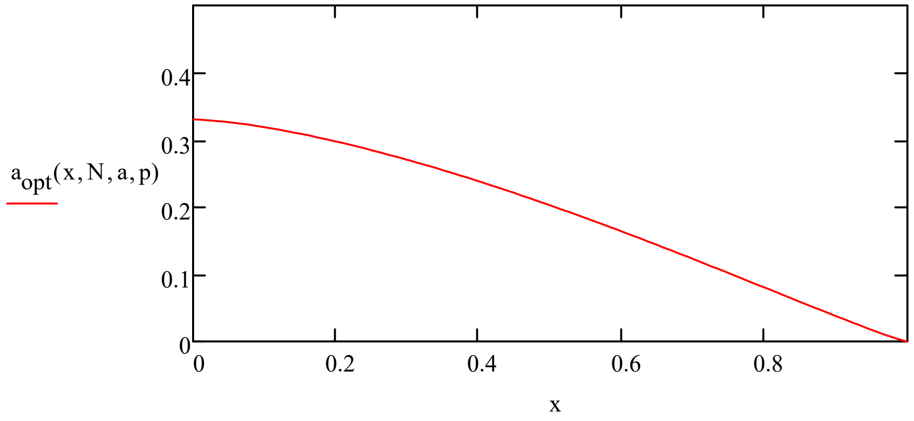


Fig2 The optimum axial induction distribution for maximum power at fixed M_0

$R_{opt}(N,a,p) = 67.133$ $\qquad \dfrac{R_{opt}(N,a,p)}{R_0} = 1.343$

$P(N,a,p) = 2.957 \times 10^{6}$ $\qquad \dfrac{P(N,a,p)}{P_0} = 1.119$

Optimum distribution of axial induction for constrained radius expansion

Given

$a \le \dfrac{1}{3}$

$R_{opt}(N,a,p) \le 1.067 \cdot R_0$

$OPT := \text{Maximize}(P,N,a,p) \qquad OPT = \begin{pmatrix} 0.417 \\ 0.333 \\ 0.136 \end{pmatrix} \qquad a := OPT_1 \qquad a = 0.333$

$N := OPT_0 \qquad N = 0.417$

$ax(x,N,a,p) := a_{opt}(x,N,a,p) \qquad p := OPT_2 \qquad p = 0.136$

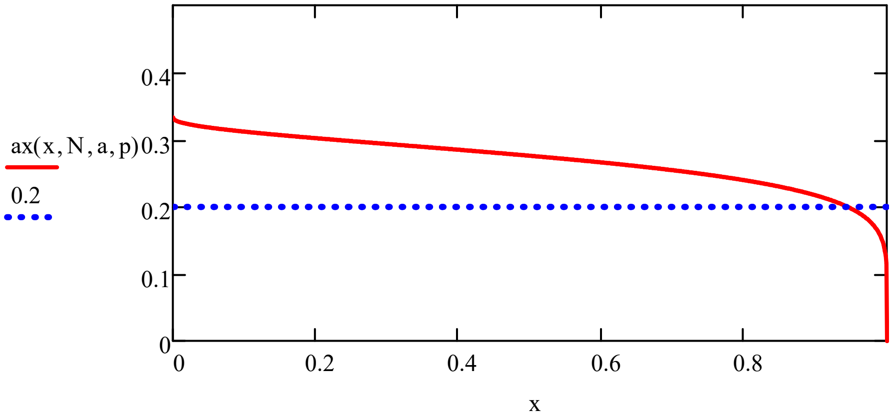


Fig3 The optimum axial induction distribution for maximum power at fixed M_0 with constrained radius expansion

$R_{opt}(N,a,p) = 53.351$ $\qquad \dfrac{R_{opt}(N,a,p)}{R_0} = 1.067$ $\qquad a = 0.333$

$N = 0.417$

$P(N,a,p) = 2.843 \times 10^{6}$ $\qquad \dfrac{P(N,a,p)}{P_0} = 1.076$ $\qquad p = 0.136$

Check that constraint on M_0 is met

$$M_0 := 0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot R_0^{\,3}\cdot Cm_r = 1.277\times 10^{7}$$

$$M_{check} := 0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot R_{opt}(N,a,p)^3\cdot\int_0^1 8\cdot a\cdot\left(1-x^N\right)^p\cdot\left[1-a\cdot\left(1-x^N\right)^p\right]\cdot x^2\cdot F(x)\,dx = 1.277\times 10^{7}$$

Bending moment distributions

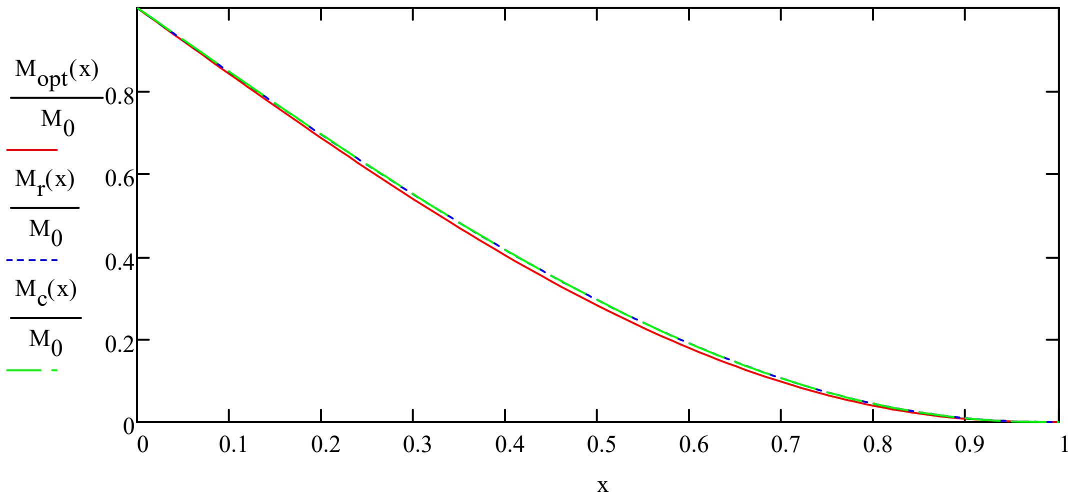
$$M_{opt}(x) := 0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot R_{opt}(N,a,p)^3\cdot\int_x^1 8\cdot a\cdot\left(1-y^N\right)^p\cdot\left[1-a\cdot\left(1-y^N\right)^p\right]\cdot(y-x)\cdot y\cdot F(y)\,dy$$

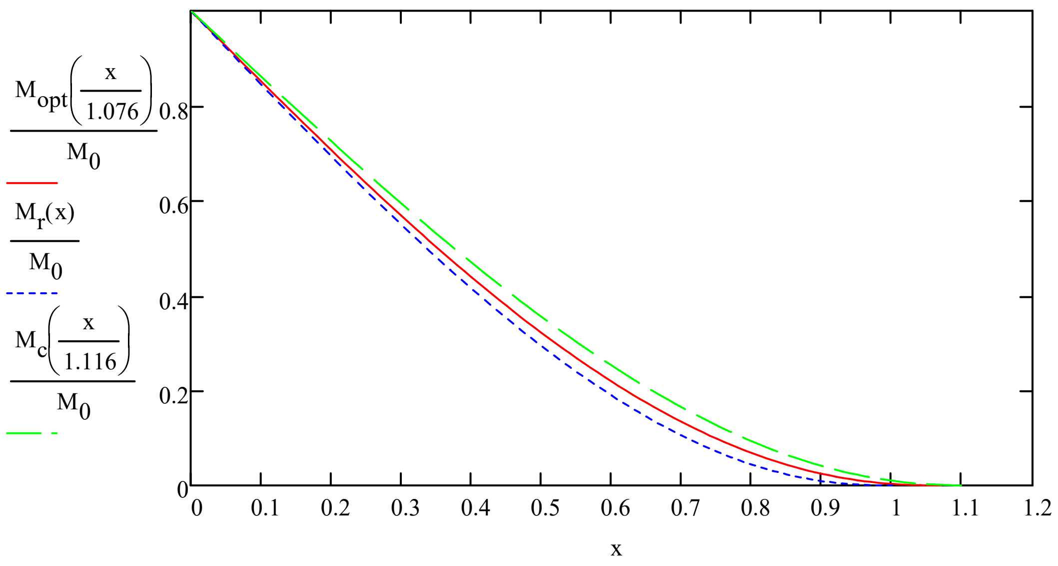
$a := \frac{1}{3}$ $\qquad$ $b := \frac{1}{5}$

$$M_r(x) := 0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot R_0^{\,3}\cdot 8\cdot a\cdot(1-a)\cdot\int_x^1 (y-x)\cdot F(y)\cdot y\,dy$$

$$M_c(x) := 0.5\cdot\rho\cdot U_0^{\,2}\cdot\pi\cdot R_0^{\,3}\cdot 1.116^3\cdot 8\cdot b\cdot(1-b)\cdot\int_x^1 (y-x)\cdot F(y)\cdot y\,dy$$

$x := 0, 0.01 .. 1.12$





$$\frac{M_{opt}\left(\frac{0.5}{1.076}\right)}{M_r(0.5)} = 1.095$$

Axial thrust force

$$Ct_{opt} := \int_0^1 8\cdot a\cdot\left(1 - x^N\right)^p\cdot\left[1 - a\cdot\left(1 - x^N\right)^p\right]\cdot x\cdot F(x)\,dx = 0.698$$

$$T_{opt} := 0.5\cdot\rho\cdot {U_0}^2\cdot\pi\cdot R_{opt}(N, a, p)^2\cdot Ct_{opt} = 3.825 \times 10^5$$

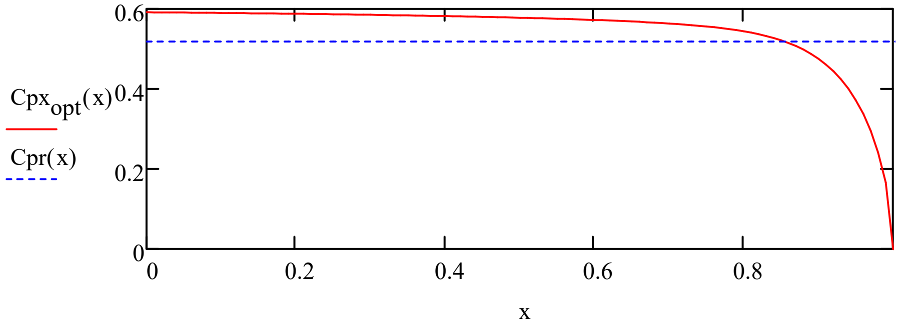
$$\frac{T_{opt}}{T_0} = 0.965$$

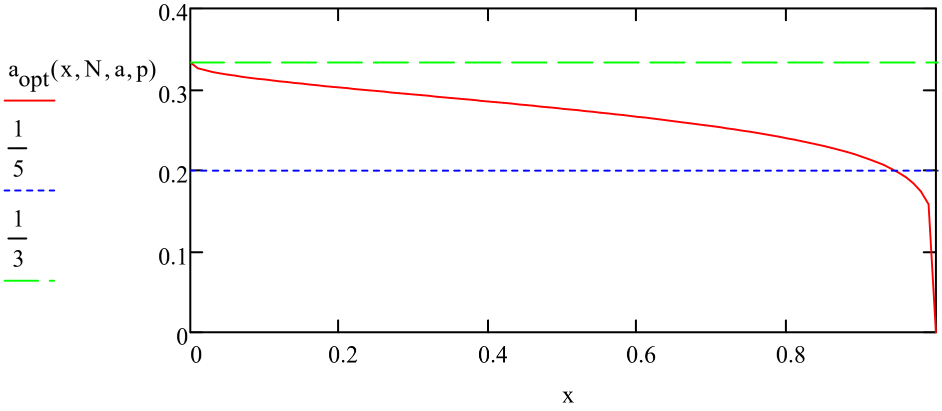
Thrust reduction is $\left(1 - \frac{T_{opt}}{T_0}\right)\cdot 100 = 3.497$ %

Power coefficient

$$Cp_{opt} := \int_{0}^{1} 8\cdot a\cdot\left(1-x^{N}\right)^{p}\cdot\left[1-a\cdot\left(1-x^{N}\right)^{p}\right]^{2}\cdot x\cdot F(x)\,dx = 0.519$$

$$Cpx_{opt}(x) := 4\cdot a\cdot\left(1-x^{N}\right)^{p}\cdot\left[1-a\cdot\left(1-x^{N}\right)^{p}\right]^{2}\cdot F(x) \qquad Cpr(x) := Cp_{opt}$$


radial distribution of Cp


radial distribution of axial induction