Appendix A: Limit values when $R_{exp} \to 2$

For the three optimization cases presented in this article (*Power-Capture, Low-Induction-Rotor, Annual-Energy-Production*, with the results summarized in table 1) the design with $R_{exp} = 2$ results in divergent values, with some of the values going towards infinity (ΔR, ΔM_{flap}, $\Delta\delta_{tip}$) while $\Delta T = 0.0\%$ and $\Delta P/AEP$ reaches a finite value. In the paper, it is not explained why this is the case, since it is not thought to be of much practical value. But for completeness, the underlying mathematical explanation for the three cases is given below.

A1 Power-Capture

Starting with the equation for the optimal C_T as a function of R_{exp} given in equation 38, and introducing $R_{exp} = 2 + \epsilon$ where ϵ is a small value, the first order expression for C_T becomes:

$$C_T = \frac{8\left((2+\epsilon)^2 - 3(2+\epsilon) + 2\right)}{(3(2+\epsilon) - 4)^2} \approx 2\epsilon \tag{A1}$$

From which it is seen that $C_T \to 0$ as $\epsilon \to 0$.

Turning to the equation for $\tilde{R}$ as a function of C_T and R_{exp} (equation 30) the equation reads:

$$\tilde{R} = \left(\frac{C_{T,0}}{C_T}\right)^{\frac{1}{R_{exp}}} \overset{C_T=2\epsilon}{\approx} \left(\frac{C_{T,0}}{2\epsilon}\right)^{\frac{1}{2+\epsilon}} \tag{A2}$$

Which for $\epsilon \to 0$ results in $\tilde{R} \to \infty$ since the exponent goes to $\frac{1}{2}$ but the denominator goes to zero.

For the Power-Capture ($\tilde{P}$) the approximate equation can be found as (using equation 35):

$$\tilde{P} \overset{C_T=2\epsilon}{\approx} \frac{C_{T,0}^{2\frac{1}{2+\epsilon}}}{2}\left(1+\sqrt{1-2\epsilon}\right)(2\epsilon)^{1-2\frac{1}{2+\epsilon}} \tag{A3}$$

The limit for $\epsilon \to 0$ is not trivial with the tricky part being $(2\epsilon)^{1-2\frac{1}{2+\epsilon}}$, but since exponents are "stronger" than the base it is found that $(2\epsilon)^{1-2\frac{1}{2+\epsilon}} \to 1$ which means that $\tilde{P} \to C_{T,0}$. It shows that even though C_T goes to zero and $\tilde{R}$ goes to infinity, $\tilde{P}$ is finite.

To compute the Δ values equation 9, 10 and 41 are used.

For ΔR it is clear that it will go to infinity since R does and R_0 is finite.

For $\tilde{P}$ it was found that $\tilde{P} = C_{T,0} = 8/9$ and since $\tilde{P}_0 = C_{P,0} = 16/27$ it shows that $\tilde{P}/\tilde{P}_0 - 1 = 1/2 = 50\%$.

The equation for ΔL is given as $\Delta L = \left(\frac{C_{T,0}}{C_T}\right)^{1-\frac{L_{exp}}{R_{exp}}} - 1$ and inserting the approximation from above it becomes $\Delta L = \left(\frac{C_{T,0}}{2\epsilon}\right)^{1-\frac{L_{exp}}{2+\epsilon}} - 1$, which results in $\Delta L \to \infty$ for cases with $L_{exp} > 2$ since the exponent is larger than zero and the denominator goes to zero. But for the case with $L_{exp} = 2$ (Thrust) $\Delta L \to 0.0\%$ since the exponent goes to zero following the same argument as for $\tilde{P}$ from before.

A2 Low-Induction-Rotor

For the Low-Induction-Rotor C_T is set to the same value as for the corresponding Power-Capture optimization for a given R_{exp}. As a consequence $C_T \rightarrow 0$ as $R_{exp} \rightarrow 2$. Investigating what happens to the rated wind speed (equation 42) in this limit gives:

$$\tilde{V}_{rated} \overset{C_T=2\epsilon}{\approx} \left(\frac{16}{27} \frac{2}{\left(1+\sqrt{1-2\epsilon}\right)2\epsilon} \left(\frac{2\epsilon}{C_{T,0}} \right)^{\frac{2}{2+\epsilon}} \right)^{\frac{1}{3-\frac{4}{2+\epsilon}}} \rightarrow \frac{16}{27} \frac{1}{C_{T,0}} = \frac{2}{3} \tag{A4}$$

Which shows that the rated wind speed is finite even in this divergent limit. Using the fact that V_0 was taken to be $10\mathrm{ms}^{-1}$ it means that $V_{rated} = 6.67\mathrm{ms}^{-1}$. This shows that there is a lower limit for the rated wind speed and it is above the Cut-In wind speed which was set at $V_{CI} = 3\mathrm{ms}^{-}1$.

The radius is computed as $\tilde{R} = \left(\frac{C_{T,0}}{C_T} \frac{1}{\tilde{V}_{rated}^2} \right)^{\frac{1}{R_{exp}}}$ and since $\tilde{V}_{rated}$ is finite it follows that also in this case $\tilde{R} \rightarrow \infty$ as $R_{exp} \rightarrow 2$. It also shows that the rotor radius is larger than the corresponding PC radius since $\tilde{V}_{rated} \leq 1$ and the LIR radius has an additional factor of $\tilde{V}^{-\frac{2}{R_{exp}}}$.

The Power-Capture ($\tilde{P}$), which is constant below rated power can be computed in the limit:

$$\tilde{P} = \frac{1}{2} \left(1 + \sqrt{1-C_T} \right) C_T \tilde{R}^2 \overset{C_T \approx 2\epsilon}{\approx} \frac{1}{2} \left(1+\sqrt{1-2\epsilon}\right) 2\epsilon \left(\frac{C_{T,0}}{2\epsilon} \right)^{\frac{2}{2+\epsilon}} \left(\frac{3}{2} \right)^{\frac{4}{2+\epsilon}} \rightarrow C_{T,0} \left(\frac{3}{2} \right)^2 = 2 \tag{A5}$$

Which means that the normalized power below rated wind speed is $\frac{27}{8}\tilde{V}^3$

Following the fact that $\tilde{R}$ and the maximum loads will happen at $\tilde{V}_{rated}$ while C_T is the same it also follows that the loads will scale in the same way as for PC, meaning that $\Delta T \rightarrow 0\%$ while the others will go towards infinity.

A3 Annual-Energy-Production

Following the results for LIR in the limit $R_{exp} \rightarrow 2$ it was found that there is a lower limit for when the rotor can reach rated power. This fact is not changed for $A\tilde{E}P$ optimization so it follows that it will have the same $\tilde{V}_{rated}$ but with additional power gained below $\tilde{V}_{rated}$ by increasing C_T to reach the constraint limit for all wind speeds in this range. The $\Delta A\tilde{E}P$ is higher than the LIR while the radius and loads behave in the same way meaning that they go towards infinity.