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Subject Author's response

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Dear Reviewers,

The authors would like to express their gratitude for the constructive feedbacks which have helped us to further improve the quality of the paper. In our attempt to accounts for the received comments, we have revised different parts of the paper. The objective of this document is to respond to the points raised by the Reviewers and to provide a detailed overview of the corresponding changes in the revised paper. In the following sections, we respond to the review report provided by each Reviewer.

Your sincerely,

Giovanni Migliaccio

Note-1: Author's response to each Referee's comment follows the comment itself and is in blue.

Note-2: At the end of the three sections above, a paper marked-up version is added (it provides a direct comparison between the revised paper and the initial paper).

Response to comments of Anonymous Referee #1

The authors are proposing a novel beam like model specifically developed for wind turbine blade structures. The authors motivate the need for development with computational efficiency required for design optimization in conjunction with aeroelastic analysis. The model is capable of considering lengthwise geometrical variations (LGVs) such as twist, curvature and pre-bend and is suitable for large deformation analysis.

General comments:
The research significance of the proposed model is high and the authors are addressing two of the renowned challenges in wind turbine blade simulations namely computational efficiency and accuracy. Regarding the latter, the implementation of LGVs into blade beam models bears indeed a considerable research demand.

- Concerning the introduction, the important contiguous contributions in the realm of this paper made by Giavotto and coworkers were not mentioned in the literature review.

  'Giavotto and coworkers' is now mentioned (see line 52 of the revised paper and the 'references').

- The model proposed in this paper is presented in a sole formal mathematical format. I am conceding the necessity of such a formal solution, albeit, the model can hardly be falsified in its current form. The authors mention that the model was indeed implemented and allude the intention to publish the procedure in a follow up paper. However, the complete absence of information concerning the implementation e.g. the pseudo code impedes reproducibility and judgement. With the information provided it is not possible to judge whether the model is a scientific breakthrough or not. In Section 4 an analytical example is presented in which no tangible results e.g. stress/strain fields are presented that would be vital for corroboration. It would especially be pertinent (and straightforward) to compare the model predictions with analytical solutions of a tapered beam the third author published previously. I recommend the paper for publication, provided that the solution is explicated in more detail with particular emphasis on the adopted numerical procedure. Moreover, the paper would gain credence by provision of concrete model predictions, which can be tried against analytical/other numerical solutions.

  An new section has been added (section 5, lines 268-367) to provide information on the current numerical implementation of the model (in Matlab), along with numerical results (e.g. tip deflections, strain measures, stress resultants) that can be obtained by using such a model (sub-sections 5.1-5.3). Comparison with corresponding results obtained with a 3D FEM commercial software are also provided.

- Specific comments/ questions:
  1. P.2 line 40: Please define 'beam like models (BLM)' or provide a reference to its stipulation

  BLM is a shorthand of "beam-like model", now it is better defined (see lines 44-46).

- 2. P.4 line 95: Please more clearly define the meaning of 'proper orthogonal tensor fields' by preferably using a physical interpretation. The same pertains to the meaning and purpose of the skew tensor fields KA and KB. Alternatively, please provide references.

Further details have been provided (see lines 100-106) and more references to classical works of rational and continuum mechanics have been added (see line 100 for 'proper orthogonal tensor…' and line 104 for 'skew tensor…').

- 3. P.5 line 110: Please more clearly enunciate the meaning of 'well-defined measures of deformation'.
  4. P.5 line 115: Please define 'proper manner'.

  That has been done and useful references have been added (see lines 118-121).

- 5. P.6 lines 150-155: The entire paragraph appears hard to follow. Can it be confate in a more comprehensible way?

  That paragraph has been revised and more details and references have been added (see lines 157-166).

- 6. P.7 top: Please clearly state which higher order terms (from which order) are neglected.

  This has been better specified (see lines 168-169).

- 7. P.7 line 170: In contrast to mathematics, I presume the majority of readers affiliated with wind energy might not be familiar with the rather specific terms stemming from differential geometry such as 'pull back' and 'push forward'. Auxiliary explanations and additional references to relevant literature would be very helpful to follow the derivation.

  This line has been re-written and additional references to classical works of rational and continuum mechanics have been provided (see lines 181-185).

- 8. The first author of one reference is misspelled: It should rather read 'Stäblein' with umlaut.

  This has been done.

- 9. P. 8 ff: Is it correct that the general beam problem is decoupled into what is stipulated as '1D' solution and into a '2D' solution? If this is indeed correctly understood, on what grounds can the decoupling be justified? What is the error estimation of such an assumption?

  For beam-like structures with transversal dimensions much smaller than the longitudinal one, in the case discussed in section 3 (small warping, small strain, etc), the resolution of the classical 3d nonlinear elasticity problem can be reduced to the resolution of two main problems. One of them governs the local warping of the cross-sections. It is referred to as the 'cross-sections problem'. The other problem governs the global deformation of the center-line. It is referred to as the 'center-line problem'. The mathematical models to determine the deformation of cross-sections and center-line are discussed with more details in the revised paper (lines 222-243 and 283-289). Additional references have been added to help understanding how those problems can be solved. An entire new section with numerical simulations has also been added (section 5) to show the accuracy of the results obtainable with such an approach and the information it can provide. Comparison with corresponding results of a 3D FEM commercial software have also been included.

- 10. P.9 line 210: If correctly understood, the 2D solution of the warping displacements must be obtained prior to the 1D solution. Yet, in equation 28 the analytical expressions for the cross sectional properties (moments of areas) of an isotropic, prismatic ellipsoid are used. It is not abundantly clear how exactly the general 6x6 cross section stiffness matrix is obtained in case of a wind turbine rotor blade.

The analytical results proposed in section 4 are for the case of tapered (not prismatic) beam-like structures with elliptical cross-sections. For that case we can provide analytical results. For generic reference cross-sections shapes the formulation of the problem of 'how to determine the deformation of the cross-sections' is the same (as in section 3.4), but in such a case the solution has to be obtained by using numerical methods. However, this is not surprising, since even in the classical linear theory of prismatic beams the analytical solutions are available for a limited number of cases only (this is better specified in the revised paper, see, for example, lines 245-249). For what concerns the relations between stress resultants and strain measures, they can be obtained by integration of the 3d stress fields over the cross-sections of the beam-like structure. In the considered case they are linear relations and can be arranged in a standard matrix form (this is better specified in the revised paper, see, for example, lines 232-237).

- 11. A figure showing the cross section, CSYS and cross-section forces used in section 4 would help a lot to illustrate the matter.

Some figures have been modified and other figures have been added to better introduce and explain the problem. In particular, see Figure 1 and Figure 2 and the corresponding lines introducing them (lines 86-95 and 142-146). They show the cross-sections and the local frames used to write the stress and strain fields, as well as the force and moments stress-resultants, in components notation. Other figures (e.g. 3, 5, 9) also help to better understand the problem and visualize the simulation results.

Response to comments of Anonymous Referee #2

The motivation of this work is highly relevant to wind energy. It is common place for beam-like models to be used, due to their balance between computational efficiency and accuracy. One limitation to these theories is the assumption of prismatic geometry. The closest example of relaxing this constraint is that of Hodges and Yu with VABS, where the beam can be curved and twisted, yet, cannot taper. Ignoring taper has some consequences for wind energy, near the root region where the loads are highest. So, the taper region can be important for structural design, while contemporary models cannot properly model these complex stresses.

- Although the ambition of this work is important to wind energy, I cannot recommend that this article is published in it's current form. A critical weakness is that the solution to the warping field is not well developed. Only a simple analytical example is given, which makes this contribution only valid for special cases. Thus, it cannot be used for wind turbine blades in general.

  The paper addresses the modeling of the mechanical behavior of beam-like structures which are curved, twisted and tapered in their reference unstressed state, undergo large displacements, in- and out-of-plane cross-sections warping and small strain. The problem of 'how to determine the warping of the cross-sections' is formulated for generic cross-sections shapes in section 3.4. For what concerns the resolution of the problem, we can provide analytical results in some cases (see section 4 for the case of bi-tapered elliptical cross-sections), while numerical methods are required for generic cases. But this is not surprising, since even in the classical linear theory of prismatic beams analytical results are available for a limited number of cases only. This is better specified in the revised paper, see, for example, lines 232-235 and 245-249.

- Currently, the state of the art are the contributions of Hodges, Yu and Giavotto. They have already developed general purpose beam models and cross section solvers. So this is the ultimate level of ambition that is needed to make a contribution to wind energy in this area. However, the key aim of this work, to incorporate taper, will be an important improvement over these earlier contributions. So I would strongly encourage the author to continue this important work. I can recognize that getting to the level of these earlier contributions will be difficult. I think this particular manuscript can still maintain an analytical approach and be improved by expanding greatly on the example. There is still an open question on what effects a beam model with taper could capture. So, the author could demonstrate the stresses and strains that this solution gives, that are not present in a more conventional beam formulation. Furthermore, the author could also make comparisons to FEM models to highlight the effects that are not captured. This I think is possible at this level and results like this would greatly improve the manuscript. Furthermore, if you had an tapered elliptical blade, how does taper affect things like frequencies or tip deflection? Again, these results will shed light on what more we can expect from simple engineering models if this limitation was relaxed, yet although simple and analytic, it would have relevance to wind energy.

  In addition to what said in the answer to the previous comment, in the revised paper a new section has been added, which includes numerical results (e.g. tip deflections, strain measures, stress resultants) for some reference beam-like structures undergoing large displacements (section 5). Different test cases confirm the effectiveness of the modeling approach and illustrate the information it can provide. Comparison with corresponding results of a 3D FEM commercial software have also been included (section 5).

- The authors did a well at explaining the motivation of their work. It could be made more widely applicable by explaining current engineering design challenges that this would help overcome. I have highlighted some points at the beginning of this review.
  This is a very mathematical paper written in a concise manner, using a lot of terminology that is typically not familiar outside of the continuum-mechanics community. To make this article accessible to wind energy readers I recommend several points where the author expand on the terminology.

  Auxiliary explanations and more references to classical works of rational and continuum mechanics have been added throughout the entire paper to make it easier to follow the mathematical aspects of the proposed modeling approach.

- The authors should further develop their techniques for solving the warping solution so it can be applied to general cross section shapes that are typically found in wind turbine blades. The authors should aim to solve the structural dynamics of real wind turbine blades. Furthermore the explanation of this work should be expanded so it is more clear.

  The method is already applicable to generic cross-sections shapes (see the answer to the first comment above, page 5 of 10).

- There are several minor points that can be improved:
  Equation 15 with sub-equations would be more clear

  The corresponding lines have been revised and further details and references have been added (see lines 168-174).

- A general comment as with a theoretical development, please elaborate on the assumptions taken and the limitations of this approach.

  Further details about the assumptions (e.g. beam-like structure, transversal dimensions much smaller than longitudinal dimension, small warping, small strain) have been provided and more references to the literature have been added (see lines 158-165).

- Generally speaking the wind energy community is not familiar with continuum mechanics. The author should explain verbally what all the terms mean. I personally have read about all these terms from my text books, but it would be nice if I didn't have to dust off my old texts to understand this article.

  Auxiliary explanations, further details on the terms used, and more references to classical works of rational and continuum mechanics, have been added throughout the paper to make it easier to follow the mathematical aspects of the proposed modeling approach.

- In the equations, the time rate of change is indicated by a dot. Typically this is given by a dot over the variable, however in this work it appears to be a super-script. This can be a little confusing because they use the same dot for dot products. If you use latex, ndot{x} would be the command that you would use.

This has been done, that is, the ‘time rate of change’ has been indicated by a dot over the variable (see, for example, Eq. 4, line 109, as well as all the other equations in which the ‘time rate of change’ is used).

- The ’ˆ’ operator is used in the equation. It is not clear that the ’ˆ’ operator is in many of the equations. The authors should elaborate more on the formal definitions of the mathematics.

The operator ’∧’ has been better defined in the revised paper (see line 106).

Response to comments of Anonymous Referee #3

The proposed method in the manuscript is a novel model of beam-like structures with curved, twisted and tapered geometries. Since the wind turbine blade designs are curved, twisted and tapered beam-like structures and go through large displacements in their operational life, the proposed model is highly related to the wind turbine blade analysis. Today, beam models are generally preferred in load and aeroelastic stability analysis of the turbine blades due to their accuracy and computational speed compared to the 3D finite element models. Although, curved and twisted beam models already exist in the literature (Hodges, Dewey H. Nonlinear composite beam theory), counting the taper effects are the main novelty of the study.

- Although the motivation of the study is very interesting and notable for state of the art blade analysis, there are essential things to be done before it is published.
  The manuscript is written in mathematical format, however the equations are hard to follow and re-derive because authors skip intermediate steps and give no reference in the derivation of the equations. I strongly recommend to write the intermediate steps explicitly or give relevant references for these steps instead of the statements such as ’well defined measures’, ’proper manner’ or ’when the 2D problem is solved’. Figures depicting the cross-sectional warping effects, loads and ’suitable coordinates’ (coordinate curves) would be helpful to the readers.

  Further details and references have been added throughout the paper to make it easier to follow the mathematical aspects of the proposed modeling approach. In addition, some figures have been modified to better introduce and explain the problem. See, for example, Figures 1 and 2. They show the center-lines in the reference and current states, the ‘plane’ cross-section in the reference state, as well as the corresponding ‘warped’ cross-section in the current state. Moreover, they also show the local frames which are used to write the stress and strain fields, the warping fields, as well as the force and moment stress-resultants, in components notation. Other figures also help to better understand the problem and visualize the simulation results (e.g. Figures 3, 5, and 9).

- Another substantial point is the lack of reproducible results. The analytical example results given in the manuscript can’t be reproduced by the explanations given in the manuscript, hence the solution needs to be explained clearly. If the authors come up with the analytical example by themselves, they should provide more information about it. If the analytical example is taken from another study, please give reference. They should also compare the their results with a higher fidelity analysis results such as 3D finite element results to show that the taper effects are captured correctly by their formulation. The authors mention that they already implemented the method in a MatLab code. However, there is no information about the implementation of the method. Example results of authors’ code and comparison of them by higher fidelity models would increase the value of the study. A wind turbine blade example would also intensify the proposed methods’ relevance to wind turbine applications.

  Section 4 provides analytical formulas we have obtained for beam-like structures with bi-tapered elliptical cross-sections. More information on this analytical solution have been provided in section 3.4 and 4. Also, an entire new section with numerical results (e.g. tip deflection, strain measures, stress resultants) for different beam-like structures has been added to show the effectiveness of the proposed approach and the information it can provide (section 5 and subsections 5.1-5.3). Comparison with corresponding results of a 3D FEM commercial software have also been included in that section.

- Please below suggestions:
  1- Section 2 : 'BeamDyn' is very relevant to the application of the geometrically exact beam models to wind turbine analysis. Consider citing it.

  It has been cited in the revised paper (see line 58 and the 'references').

- 2- Section 3.1 : Instead of Figure-2 with wind turbine blade, a figure with cross-section warpings and coordinate curves would be elucidating.

  Some figures have been modified and other figures have been added to better introduce and explain the problem. In particular, Figure 1 and Figure 2 have been modified, as well as the corresponding lines introducing them (lines 86-95 and 142-146). They show the center-lines in the reference and current states, the 'plane' cross-section in the reference state, as well as the corresponding 'warped' cross-section in the current state. Moreover, they also show the local frames which are used to write the stress and strain fields, the warping fields, and the stress-resultants, in components notation.

- 3- Section 3.1 : Please explain 'y' clearly (in current position vector R).
  4- Section 3.1 : Please explain deformation gradient explicitly or give reference for it.
  5- Section 3.1 : Please explain 'some higher terms' after equation 14.
  6- Section 3.1 : Please write intermediate steps between equation 13 - 15.

  The paragraph containing 'y', 'deformation gradient', 'higher order terms' and 'equation 13-15' has been modified and more explanations have been added. Further details on the mathematical model have been provided, along with more references to classical works of rational and continuum mechanics. See lines 149-174.

- 7- Section 3.2 : A figure with cross-section forces and moments would help the readers.

  Some figures have been modified and other figures have been added to better introduce and explain the problem, as mentioned above. See Figures 1-2 and the corresponding lines introducing them. They show the cross-sections and also the local frames which are used to write the stress and strain fields, the warping fields, and the force and moment stress-resultants too, in components notation.

- 8- Section 3.4 : Please elaborate the section by providing the solution of the warping fields.

  Section 3.4 addresses the problem of 'how to determine the warping fields', which are responsible of the cross-sections deformation. Section 4 provides analytical results for beam-like structures with bi-tapered elliptical cross-sections. For that case we can provide analytical results. For generic cross-sections shapes the formulation of the problem is the same as in section 3.4, but the solution has to be obtained by using numerical methods. But this is not surprising, since even in the classical linear theory of prismatic beams analytical solutions are available for a limited number of cases only. This has been better specified in the revised paper (see also lines 232-235 and 245-249).

- 9- Section 4 : Please give more information about the example and how you obtain the

final results. Please, compare them with higher fidelity solution to show your model captures the taper effects correctly. Comparison can also show the results of a model which ignores the taper effects. So, reader can see the effect of taper term in final results.
10- A section which explains the numerical implementation should be added.
11- A section with results of your numerical model and higher fidelity model should be added.

Further details on how to determine cross-sections warping and center-line deformation have been provided (see lines 222-243 and 283-289). An entire new section has been added to introduce the model we have implemented in Matlab and the results that it can provide (see section 5). Comparison with corresponding results of a 3D FEM commercial software have also been included in that section.

Beam-like models for the analyses of curved, twisted and tapered HAWT blades in large displacements

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Abstract. The continuous effort to better predict the mechanical behavior of complex beam-like structures like wind turbine blades is strictly related to ~~lowering the cost~~requirements of ~~energy.~~performance improvement and costs reduction. But new design ~~strategies and the continuous increase in the size~~approaches and the increasing flexibility of ~~modern blades~~those structures make their aero-elastic modeling ever more challenging. For the structural part of this modeling, the best compromise between computational efficiency and accuracy can be obtained by ~~schematizing the blades as~~a schematization based on suitable beam-like elements. This paper addresses the modeling of the mechanical behavior of ~~complex~~ beam-like structures~~,~~ which are curved, twisted and tapered in their reference unstressed state, undergo large displacements, ~~3D~~in- and out-of-plane cross-sectional warping and small strains. A suitable model for the problem at hand is proposed.~~ It can be used to analyze large deflections under prescribed loads and allows the 3D strain and stress fields in the structure to be determined.~~ Analytical and numerical results obtained by applying the proposed modeling approach, as well as comparison with 3D-FEM results, are illustrated.

1 Introduction

In the process of improving horizontal axis wind turbines (HAWT) performance new methods are continuously being sought for capturing more energy~~ and~~, developing more reliable structures, ~~all with the ultimate goal of~~and lowering the cost of energy (Wiser, 2016).~~ As demonstrated by several researches, such goal~~ Such goals can be achieved through the use of advanced materials, the optimization of the aerodynamic and structural behavior of the blades, and the exploitation of load control techniques~~. By way of~~ (see, for example, ~~one promising load control approach is based on the bend-twist coupling (BTC) of the blades, which can be obtained by sweeping the shape of the blades or by changing the orientation of their composite fibers (~~Ashwill 2010, Bottasso 2012, ~~Stablein~~Stäblein 2017). But new design strategies and the ~~continuous increase in the size and~~increasing flexibility of ~~modern blades~~those structures make the modeling of their aero-elastic behavior ever more challenging. For the structural part of this modeling, schematizing the ~~blades as~~baldes through suitable beam-like elements can be the best compromise between computational efficiency and accuracy. ~~Modern~~But modern blades ~~can be considered~~ are complex beam-like structures~~, which are~~ . They can be curved, twisted and also tapered in their

~~reference~~ unstressed state. Even not considering the complexities related to the materials ~~properties~~ and ~~the actual~~ loading conditions, their shape alone is sufficient to make the mathematical description of their mechanical behavior a ~~challenging task~~very challenging task. This paper addresses the modeling of the mechanical behaviour of structures of this kind, with a particular focus on their main geometrical characteristics, such as the twist and taper of their cross-sections, the in- and out-of-plane warping of their cross-sections, and the large deflections of their reference center-line.

~~The present work addresses the mechanical modeling of modern blades considered as suitable curved, twisted and tapered~~ Over the years several theories have been developed for beam-like structures~~. Beam~~ (see, for example, Love 1944, Antmann 1966, and Rubin 1997). This is because beam models have historically ~~found application~~been used in many fields, from ~~the~~ helicopter rotor blades in aerospace engineering~~,~~ to bridges components in civil engineering~~,~~ and surgical tools in medicine. ~~This contributed to the development of sophisticated theories over the years (see, for example, Love 1944, Antmann 1966, Rubin 1997). The need for geometrically non-linear models~~ Nevertheless, the interest in advanced theories for complex beam-like structures has led to further researches also in recent years~~. One of the main drivers for~~, due to the continuous ~~research in this field is the~~ need ~~for~~of ever more rigorous and application-oriented models. In this paper the attention is focused on the effects of important geometrical ~~design features~~characteristics of those structures, such as the ~~curvatures~~curvature of ~~the reference~~their center-line~~ and~~, as well as the twist and the taper of ~~the~~their cross-sections. After an introduction to modeling approaches for structures of this kind (section 2), a suitable model for the problem at hand is proposed (section 3). Finally, ~~some~~ analytical results and numerical examples obtained by applying the proposed modeling approach to reference beam-like structures are illustrated (~~section~~sections 4 and 5).

2 Overview of modeling approaches

Aero-elastic modeling of modern blades can be addressed by means of different approaches (Wang ~~2016~~2016a). Those ones based on 3D FEM and beam~~-like~~ models ~~(BLM)~~ are two main choices for the structural part of this modeling. Although 3D FEM approaches can be very accurate and flexible, they can be computationally expensive for the analyses of complex systems, especially if CFD ~~aerodynamic~~ analyses are executed in parallel. The overall computational cost can be reduced ~~by using~~if faster aerodynamic models are used, such as the blade element momentum (BEM) model, but even this solution may not be efficient enough for aero-elastic analyses and multi-objective optimization tasks. The coupling of ~~BLM~~ BEM and ~~BEM~~ suitable beam models can ~~provide~~be the best compromise between computational efficiency and accuracy. In this work we focus the attention on ~~BLM for~~ mathematical models to simulate the mechanical behaviour of complex beam-like structures~~, such~~ (hereafter referred to as ~~modern blades,~~beam-like models, or BLM), which can be curved, twisted and also tapered in their unstressed state, ~~be subjected to~~undergo large deflections, in- and ~~3D~~out-of-plane cross-sectional ~~warpings.~~warping and small strain. Suitable models are needed to simulate ~~their mechanical behavior.~~the behaviour of those structures. In general, classical beam models (see, for example, Love 1944), which include extension, twist and bending, as well as the Reissner's formulation~~ of Reissner~~ (1981), also accounting for transverse shear deformation, may not be sufficient. Geometrically exact models are a better choice, but a way to put them into a suitable form for engineering applications is usually needed (Antman 1966). In general, suitable models should be both rigorous and application-oriented, two important requirements pursued over the years by many investigators (~~see, for example~~e.g. Giavotto 1983, Simo 1985, Ibrahimbegovic 1995, Ruta 2006, Pai 2011, and Yu 2012).

For over a century researchers have sought to represent beam-like structures by means of 1D models. Several theories have been developed, from the elementary Euler-Bernoulli theory, to the classical theory which includes Saint-Venant torsion, up to more refined theories, such as the Timoshenko theory for transverse shear deformations, the Vlasov theory for torsional warping restraint, and 3D beam theories which include 3D warping fields. Broadly speaking, beam theories can be grouped into engineering and mathematical theories. ~~Among the~~Several engineering theories~~, some formulations~~ are based on ad-hoc corrections to simpler theories (e.g. Rosen 1978), ~~while~~ others are based on geometrically exact approaches (such as Wang 2016b and Hodges 2018). Among the mathematical theories, some approaches are based on the directed continuum (Rubin 1997), some others exploit asymptotic methods (Yu 2012). The reason for ~~the extensive and continuous research efforts~~such a large amount of works on beam theory is that it has always found ~~applications~~application in many fields. ~~For~~By way of example, many approaches have been developed for helicopter rotor blades with an initial twist~~. Pre~~, but pre-twisted rods have always attracted the interest of many researchers in ~~different~~several fields~~. A wide-ranging review on this subject is due to~~ (Rosen ~~(~~1991). In the 1990's, Kunz (1994) provided an overview~~ also~~ on modeling methods for rotating beams, illustrating how engineering theories for rotor blades evolved over the years. In those same years, Hodges (1990) reviewed the modeling approaches for composite rotor blades, discussing the importance of 3D warping and deformation coupling. More recently, Rafiee (2017) discussed vibrations control issues ~~for~~in rotating beams, summarizing beam theories and complicating effects, such as non-uniform cross-sections, initial curvatures, twist and sweep. It seems that, unlike the case of the pre-twisted rods, ~~the~~ published results for curved rotating beams with initial taper and sweep are quite scarce, although all these geometrical characteristics can play an important role.

Up to now much has been done to develop powerful beam theories. However, there is still a gap between existing theories and those that could be suitable for complex beam-like structures. In general, the geometry of the reference and current states must be appropriately described. The curvature, twist and taper are important design features and should be explicitly included in the model. The analysis should not be restricted to small displacements~~ and should consider deformation couplings.~~. The model should provide the strain and stress fields in the three-dimensional beam-like structure, be rigorous and usable by engineers, and provide classical results when applied to prismatic isotropic homogeneous beams. Following these guidelines, a mathematical model to simulate the mechanical behavior of ~~complex~~the considered beam-like structures is proposed hereafter.

3 Mechanical model for complex beam-like structures

Here we are concerned with developing a mathematical model to describe the mechanical behavior of beam-like structures which are curved, twisted and tapered in their reference state and undergo large displacements. One of the main issues with such a task is how to describe the motion of the structure (see, for example, Simo 1985, Ruta 2006, ~~and~~ Pai 2014). The approach considered in this work is to imagine a beam-like structure as a collection of plane figures (i.e. the cross-sections) along a regular and simple three-dimensional curve (i.e. the center-line). We assume that each point of each cross-section in the reference state moves to a position in the current state through a global rigid motion on which a local general motion is superimposed. In this manner, the cross-sectional deformation can be examined independently of the global motion of the center-line. So, it is possible to consider the global motion to be large, while the local motion and the strain may be small.

3.1 Kinematics and strain measures

We begin by introducing two local triads of orthogonal unit vectors. The first one is the local triad, b_i, in the reference state, with b_1 aligned to the tangent vector of the reference center-line. This frame is a function of the reference arch-length ~~parameter only~~s, that is $b_i=b_i(s)$. The second local triad, a_i, is a suitable image of the local triad b_i in the current state. This frame is a function of the reference arch-length ~~parameter~~s and the time t, that is $a_i=a_i(s,t)$. In general, a_1 is not required to be aligned to the tangent vector of the current center-line. See Figure 1. It shows a schematic representation of the reference (left) and current (right) states of a beam-like structure. The generic cross-section in the reference state is contained in the plane of the vectors b_2 and b_3. In the current state, if the cross-section remains plane (i.e. un-warped), it can belong to the plane of the vectors a_2 and a_3. But the generic cross-section may not remain plane. So, we consider that its current (warped) state is reached by superimposing an additional motion to the positions of the points of the un-warped cross-section, as in Figure 1 (right).

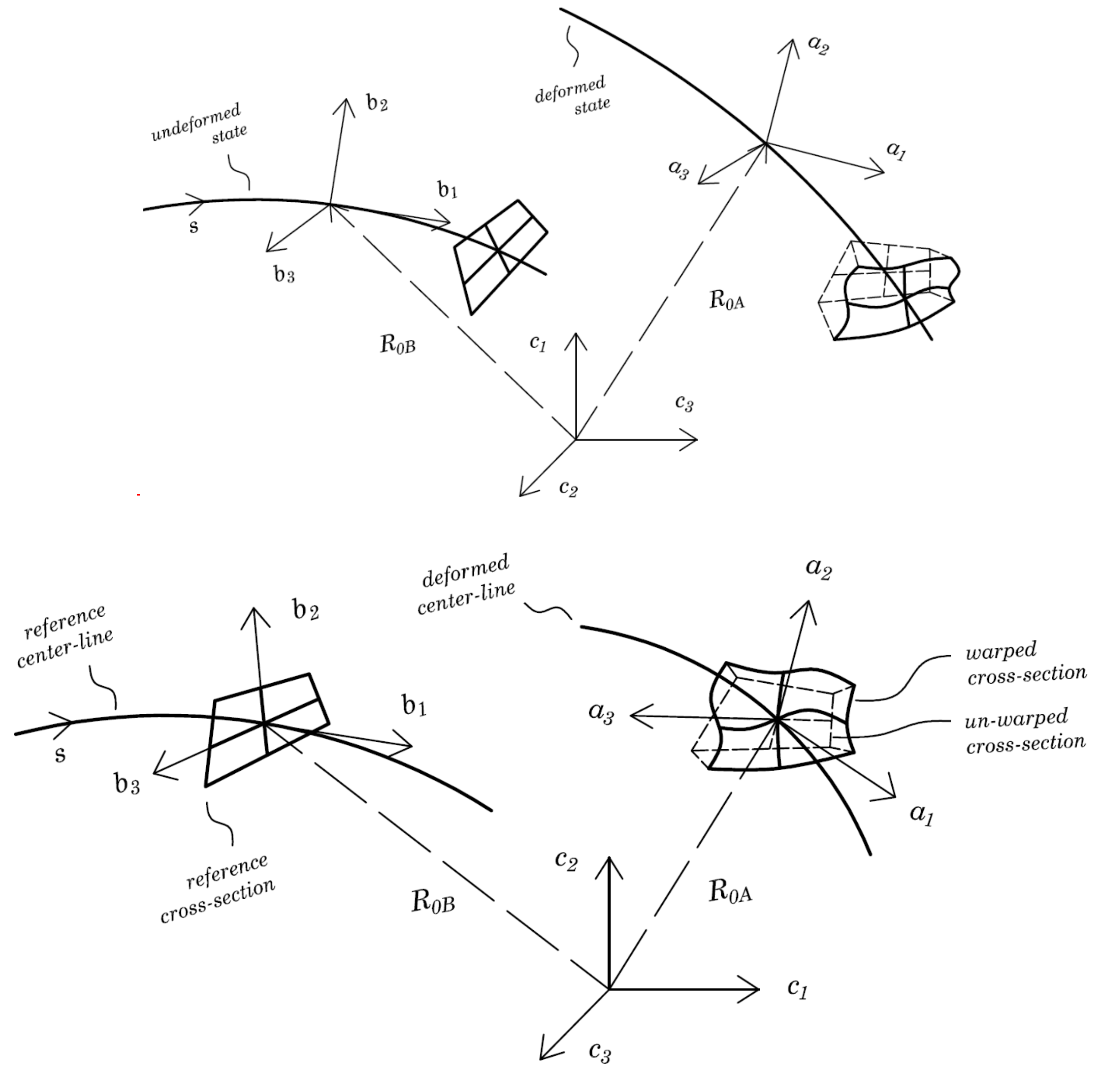

Figure 1: Schematic of the reference and current states, center-lines, cross-sections and local frames

We continue by introducing the kinematical variables we use to describe the motion of the considered structure. To this aim, the orientation of the ~~frame~~frames a_i and b_i relative to a fixed rectangular frame, c_i, are defined as follows

$$a_i = Ac_i, \quad b_i = Bc_i \tag{1}$$

where A and B are two proper orthogonal tensor fields~~.~~ (i.e. their determinant is 1, see, for example, Gurtin 2003). We also introduce ~~an orthogonal~~a tensor field, T, which defines the relative orientation between the frames a_i and b_i ~~and can be used to identify the deformed configuration of the structure, as~~as follows

$$a_i = Tb_i = AB^T b_i \tag{2}$$

We ~~also~~ define two skew tensor fields, K_A and K_B, and their axial vectors, k_A and k_B, which are related to the curvatures of the center-line of the structure~~, respectively~~ in the current and reference states, as follows (see, for example, Simo 1985 and Gurtin 2003)

$$\begin{aligned} K_A &= A^{'} A^T, \quad a_i^{'} = K_A a_i = k_A \wedge a_i \\ K_B &= B^{'} B^T, \quad b_i^{'} = K_B b_i = k_B \wedge b_i \end{aligned} \tag{3}$$

~~The~~where the prime denotes derivative with respect to the arch-length ~~parameter~~, s~~. Then~~, while the operator ∧ is the usual cross-product. In a similar manner, we introduce the skew tensor field Ω, and its axial vector field ω, ~~associated with~~related to the variation of the vectors a_i over the time, t, as follows

$$\Omega = \dot{A} A^T, \quad \dot{a}_i = \Omega a_i = \omega \wedge a_i \tag{4}$$

The dot (over the variables) denotes derivative over the time, t. At this point, it is easy to obtain the following identities

$$\begin{aligned} T^{'} T^T &= K_A - T K_B T^T, \quad \dot{T} T^T = \Omega \\ \phi\left[T^{'} T^T\right] &= k_A - T k_B, \quad \phi\left[\dot{T} T^T\right] = \omega \end{aligned} \tag{5}$$

where the operator ϕ[] provides the axial vector of the skew tensor between brackets.

The function R_{0B}, which maps the points of the center-line in the reference state, does not depend on time, while R_{0A} can change over the time t. Its variation is the time rate of change of the position of the points of the current center-line

$$\dot{R}_{0A} = v_0 \tag{6}$$

We are now in a position to introduce two important kinematic identities

$$\begin{aligned} v_0^{'} - \omega \wedge R_{0A}^{'} &= T \dot{\gamma} \\ \omega^{'} &= T \dot{k} \end{aligned} \tag{7}$$

where γ and k are ~~well-defined measures of deformation for beam-like structures. They vanish for pure rigid motions and transform in the proper manner when a rigid motion is superposed to a not rigid motion. They are defined as~~

$$\begin{aligned} \gamma &= T^T R_{0A}^{'} - R_{0B}^{'} \\ k &= T^T k_A - k_B \end{aligned} \tag{8}$$

It is worth nothing that γ and k vanish for rigid motions and are invariant under superposed rigid motion, hence, they are well-defined measures of strain for beam-like structures (see, for example, Ruta 2006 and Rubin 2000).

Now~~,~~ we start modeling the motion of the points of the cross-sections. In particular, we introduce two mapping functions, R_A and R_B, to identify the positions of the points of the 3D beam-like structure in its current and reference states. For what the reference state is concerned, we define the (reference) mapping function

$$R_B(z_i) = R_{0B}(z_1) + x_\alpha(z_i) b_\alpha(z_1) \tag{9}$$

where R_{0B} is the position of the points of the reference center-line relative to the frame c_i, b_α are the vectors of ~~such~~the reference local frame in the plane of the reference cross-section, x_α identify the position of the points in the reference cross-section relative to the reference center-line, and, finally, z_i are ~~suitable coordinates which do not depend on time, with z_1=s.~~ independent mathematical variables which do not depend on time. In particular, z_1 is equal to the arch-length s, and z_α belong to a bi-dimensional mathematical domain which is used to map the position of the points, x_α, of the cross-sections. Throughout this paper, Greek indices assume values 2 and 3, Latin indices assumes values 1, 2 and 3, and repeated indices are summed over their range.

It is worth noting that x_k may or may not be equal to z_k~~, with the~~. The first choice ~~leading~~leads to the most common modeling approaches available in the literature (see, for example, Simo 1985, Pai 2011, and Yu 2012). In this work we ~~use~~choose a ~~different approach, by choosing suitable~~ set of relations between the position variables x_k and the mathematical variables z_k to ~~simulate~~provide a description of the shape of the considered beam-like structure, which ~~is~~can be curved, twisted and also tapered in its reference ~~unstressed~~ state. In particular, the span-wise variation of the shape of the cross-sections is analytically modeled by means of a mapping of this kind

$$x_i = \Lambda_{Bij} z_j \tag{10}$$

where the coefficients Λ_{Bij} are ~~suitable~~ functions of z_1. In the following we will consider the ~~interesting class~~case of the curved and twisted beam-like structures with bi-tapered cross-sections, in which case the map ~~of Eq.~~ (10) reduces to

$$x_1 = z_1, \quad x_2 = z_2 \lambda_{x2}(z_1), \quad x_3 = z_3 \lambda_{x3}(z_1) \tag{11}$$

where the coefficients $\lambda_{x\alpha}$ are ~~suitable~~ functions of z_1. It is worth noting that a suitable ~~definition~~choice of ~~such~~those functions gives the possibility to reproduce interesting shapes~~, such as~~. See, for example, Figure 2. It shows a 3D beam-like structure whose center-line is curved, while the ~~one reported~~cross-sections are twisted and tapered from the root to tip. The reference cross-sections in Figure 2~~.~~ are ellipses with different sizes and orientations, but any other reference cross-section shape can be considered, such as the aerodynamic profiles which are commonly used for wind turbine blades, steam turbines blades, and helicopter rotor blades as well (see also Griffith 2011, Bak 2013, Tanuma 2017, and Leishmain 2006 for examples of such profiles).

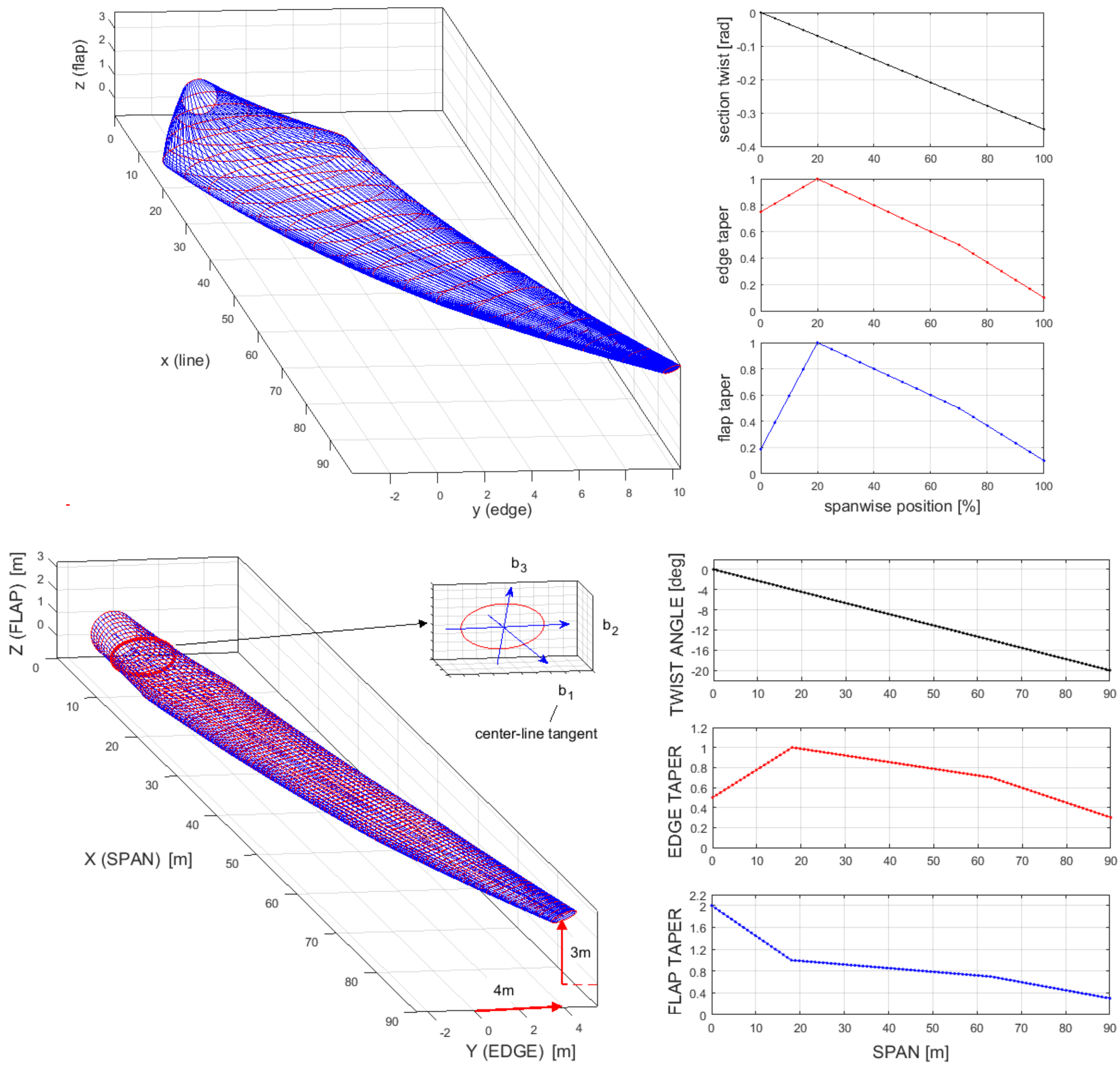

Figure 2: ~~Schematic~~Example of ~~a~~ curved, twisted, and tapered beam-like structure and local frame (left), taper and twist functions (right)

The position of the points in the current state are defined in a similar manner by means of the (current) mapping function

$$R_A(z_i,t) = R_{0A}(z_1,t) + x_\alpha(z_i)a_\alpha(z_1,t) + w_k(z_i,t)a_k(z_1,t) \tag{12}$$

where R_{0A} is a function mapping the position of the points of the center-line in the current state, while w_k are the components of the 3D warping displacements in the local frame a_k. ~~Again, y_k is not equal to z_k. In general, we reserve the possibility to~~

~~choose relations similar to Eq. (10), even with coefficients A_{Aij} (and $\lambda_{y\alpha}$) different from A_{Bij} (and $\lambda_{x\alpha}$).~~ The main formal difference between the reference and current maps is due to the warping, w, introduced to describe the geometry of the deformed state without a-priori approximation.

~~The~~By using the maps (9) and (12), we can determine the 3D ~~deformation~~tensor, H, expressing the gradient~~, H, between~~ of the current position, R_A, with respect to the reference ~~and current configurations, can now be calculated~~position, R_B, as follows (see, for example, Rubin 2000)

$$H = \frac{\partial R_A}{\partial R_B} \tag{13}$$

~~where G_k and g_k are the covariant and controvariant base vectors in the current and reference states, respectively. They can be determined by using standard means.~~ When ~~the deformation gradient~~H is ~~given~~known, the 3D Green-Lagrange strain tensor, E, can be calculated~~. In particular, we~~ (see, for example, Rubin 2000 and Gurtin 2003). Hereafter we write the ~~Green-Lagrange strain~~ tensor E in a form based on simplifying assumptions applicable to the considered beam-like structure. ~~To this end~~In particular, we introduce the characteristic dimension of the cross-sections, herein denoted as h, the longitudinal dimension of the ~~reference~~ center-line, herein denoted as L, and we assume h to be much smaller than L. ~~Considering~~Moreover, we consider a thin structure~~, we~~ and assume ~~its~~ the curvatures of its reference center-line are much smaller than 1/h~~. The~~ (see also Rubin 2000). In addition, we assume the warping displacements, w_k, are ~~also assumed to be~~ small. More precisely, by introducing a non-dimensional parameter ε much smaller than one, they are considered of the order of hε, while the order of magnitude of their derivative with respect to z_1 is ~~of the order of~~ εh/L. In ~~addition~~general, all ~~deformations~~deformation measures, i.e. the 1D strain measures γ and k and the components of the 3D strain tensor, E, are assumed to be small. In particular, ~~γ is much smaller than one and kh is of the~~ their order of magnitude is at most ε.

For the considered structure, in the case of small strains and small local rotations, ~~the~~ we write the strain tensor, E, in the following ~~relation holds~~form

$$E \simeq \frac{T^T H + H^T T}{2} - I \tag{14}$$

~~By way of example, in the case of uniform initial taper ($A_{B22}=A_{B33}=A_0$), $y_k=x_k$, and neglecting some higher order terms, the components of the Green-Lagrange strain tensor may be written in the form~~

Let's now calculate the components of E by using (14) and neglecting terms smaller than ε. Algebraic manipulations only, too lengthy to be reported here in full, yields the following expressions for bi-tapered cross-sections

$$
\begin{aligned}
&E_{11} = \gamma_1 + k_2 \Lambda_{B33} z_3 - k_3 \Lambda_{B22} z_2 \\
&E_{22} = \Lambda_{B22}^{-1} w_{2,2} \\
&E_{33} = \Lambda_{B33}^{-1} w_{3,3} \\
&2E_{21} = \gamma_2 + \Lambda_{B22}^{-1} w_{1,2} - k_1 \Lambda_{B33} z_3 \\
&2E_{31} = \gamma_3 + \Lambda_{B33}^{-1} w_{1,3} + k_1 \Lambda_{B22} z_2 \\
&2E_{23} = \Lambda_{B33}^{-1} w_{2,3} + \Lambda_{B22}^{-1} w_{2,2}
\end{aligned}
\tag{15}
$$

~~where~~

In (15) Λ_{B22} and Λ_{B33} are the edge-wise and flap-wise taper coefficients (see, for example, Figure 2), while the components of the strain tensor, E, are taken with respect to the reference local frame, b_i, i.e.

$$E_{ij} = E \cdot b_i \otimes b_j \tag{16}$$

where $\cdot$ is the usual scalar (or dot) product and $\otimes$ is the tensor (or dyadic) product (see, for example, Rubin 2000).

3.2 Stress fields and constitutive models

Given the strain tensor, E, the stress fields in the structure can be calculated when a constitutive model is chosen. Limiting our attention to elastic bodies, in a pure mechanical theory, in the case of small strain, we use ~~a~~the following linear relation between the second Piola-Kirchhoff stress tensor, S, and the Green-Lagrange strain tensor (see, for example, Gurtin 2003~~), as follows~~)

$$S = 2\mu\, E + \lambda tr E\, I \tag{17}$$

where μ and λ are known material parameters related to the Young's modulus and Poisson's ratio.

In the case of small ~~strain~~strains and small local rotations, we ~~can~~ also write

$$P \simeq TS, \quad C \simeq TST^T \tag{18}$$

where P is the first Piola-Kirchhoff stress tensor and C is the Cauchy stress tensor~~. In~~ (Gurtin 2003). It is worth noting that in the considered case the tensor field T is sufficient to ~~perform~~determine two of the ~~pull back and push forward operations between the~~ above mentioned stress ~~tensor fields S,~~tensors (e.g. P and C~~.~~) when the other one (e.g. S) is known.

~~Now, we~~We are now in the position to define the cross-sectional stress resultants, namely the force F and moment M~~, on each cross-section of the structure.~~. Using the first Piola-Kirchhoff stress tensor~~,~~ (Gurtin 2003), in the case of small warpings, small strains and small local rotations, we write

$$F = T\int_{\Sigma} P_{i1} b_i, \quad M = T\int_{\Sigma} x_\alpha P_{i1}\, b_\alpha \wedge b_i \tag{19}$$

where Σ is the domain corresponding to the cross-section on which the integration is performed and

$$P_{ij} = P \cdot a_i \otimes b_j \tag{20}$$

By combining Eqs. (15)-(19), the force and moment stress resultants can be related to the geometrical parameters of the structure and the 1D strain measures (8). However, such relations are actually known if we know the warping fields w_k. An approach to obtain suitable warping fields is illustrated in section 3.4.

3.3 Expended power and balance equations

To complete the formulation, we conclude with considerations on the principle of expended power and the balance equations for the considered structure. For hyper-elastic bodies (Gurtin 2003), we write the principle of expended power in the form

$$\int_A p\cdot v+\int_V b\cdot v=\frac{d}{dt}\int_V \Phi \tag{21}$$

where p are surface loads per unit reference surface (A), b are body loads per unit reference volume (V), Φ is the 3D energy density function of the body, and v is the time rate of change of the current position of its points, which is given by

$$v=v_0+\omega\wedge y_\alpha a_\alpha+\dot{w} \tag{22}$$

where ~~w•~~the last term in (22) is the time rate of change of the warping displacement.

For small warpings, small strains, and small local rotations, ~~it can be shown that~~if the power expended by ~~the~~ surface and body loads on the warping velocities ~~can be~~is neglected~~. By using this simplification~~, the external power, Π_e, reduces to the following form

$$\Pi_e=\Delta\left(F\cdot v_0+M\cdot\omega\right)+\int_s F_s\cdot v_0+M_s\cdot\omega \tag{23}$$

where the vector field v_0 is the time rate of change of the position of the points of the current center-line, the vector field ω is the time rate of change of the orientation of the vectors a_i, the terms F_s and M_s are suitable resultants of inertial actions and prescribed loads per unit length in the reference state, while the symbol Δ simply means that the function between brackets is evaluated at both the ends of the beam and the difference between those values is taken.

The ~~3D~~ cross-sectional warpings may be important in calculating the 3D energy function~~, so they~~ and cannot be neglected in the internal power, Π_i. However, the internal power may be reduced to a useful form for beam-like structures by introducing a suitable 1D strain energy function, U. For example, if U can be expressed in terms of the strain measures, γ and k, we ~~obtain~~can write

$$\Pi_i=\frac{d}{dt}\int_s U(\gamma,k,s)=\int_s f\cdot\dot{\gamma}+m\cdot\dot{k} \tag{24}$$

where ~~f and m are~~ the ~~pulled back~~ vector fields f and m are defined in terms of the force and moment stress resultants, F and M, ~~and are defined~~ as follows

$$f=T^T F,\quad m=T^T M \tag{25}$$

By using the principle of expended power, we also obtain balance equations for the vector fields F and M in the form

$$\begin{aligned} F^{'} + F_s &= 0 \\ M^{'} + R_{0A}^{'} \wedge F + M_s &= 0 \end{aligned} \tag{26}$$

At this point, we have kinematic equations, (6)-(7), strain measures, (8) and (14), force and moment balance equations, (26), and the principle of expended power, $\Pi_e=\Pi_i$, in a suitable form for beam-like structures, (23)-(24). To complete the formulation of the model we need relations providing the 1D stress resultants in terms of the 1D strain measures. To this end, we need to know the warping fields. An approach to obtain suitable warping functions is discussed hereafter.

3.4 Warping displacements

In general, a 3D nonlinear elasticity problem can be formulated as a variational problem. In any case, if we try to solve the variational problem directly, the difficulties encountered in solving the elasticity problem remain. For beam-like structures whose transversal dimensions are much smaller than the longitudinal one, assumptions based on the shape of the structure and the smallness of the warping and strain fields can lead to ~~important~~useful simplifications. In particular, the resolution of the 3D nonlinear elasticity problem can be ~~split into a 1D nonlinear~~reduced to the resolution of two main problems. See, for example, Berdichevsky (1981), who seems to be the first in the literature to plainly state that for elastic rods. One of those problems governs the local distortion of the cross-sections and is here referred to as the cross-sections problem~~, governing~~. The other problem governs the global deformation of the center-line and ~~cross-sectional frames, and a 2D problem, governing the local distortion of the cross-sections. The warping displacements can be obtained by solving the 2D~~is here referred to as the center-line problem. ~~Using such an approach, for small warpings, strain and local rotations,~~Hereafter, we consider the following variational statement ~~can be used~~to determine the warping fields which are responsible of the deformation of the cross-sections

$$\delta \int_V \Phi = 0 \tag{27}$$

~~where~~In (27) the symbol δ ~~is~~ stands for the variation ~~of~~operator and the ~~functional for a corresponding variation of~~density function Φ depends on the warping ~~fields.~~

displacements. The warping fields satisfying (27) can be obtained by the corresponding Euler-Lagrange equations~~. When the 2D~~ (see, for example, Courant 1953), by using numerical methods, in general, or analytical approaches providing closed-form expressions, in some particular cases. Once such a problem is solved, the ~~stiffness properties~~components of the ~~cross-sections are known. From~~stress resultants (19) can be linearly related to the ~~1D problem we obtain~~components of the 1D strain measures ~~(8). Finally, given the warping fields and the 1D strain measures, the 3D strain and stress fields can be determined (14)-(18).~~, by using equations (14)-(19). Then, if it is preferred or deemed useful, those relations can also be arranged in a standard matrix form.

Note that to determine the current state of the structure we also need the displacements of its center-line points. They can be determined by solving the center-line problem, which is a non-linear problem governed by the set of kinematic, constitutive and balance equations introduced in section 3 (in particular, we are referring to the constitutive model in section 3.2, which relates stress resultants and strain measures, and the balance equations for the stress resultants in section 3.3).

In the next sections we show some analytical solutions (section 4 ~~An example with analytical~~) and numerical results

~~An example illustrates the results~~ (section 5) that can be obtained by applying the proposed modeling approach ~~can provide.~~ to some reference beam-like structures.

4 First analytical results for bi-tapered cross-sections

In ~~particular,~~this section we consider ~~a curved, twisted and tapered~~ the case of a beam-like structure with bi-tapered elliptical cross-sections. ~~The structure is clamped at one end and it is loaded by given forces at the other end. We use the assumptions introduced in the foregoing about~~For this case we can provide analytical solutions in terms of warping fields, while for generic shapes (e.g. the aerodynamic profiles used in wind turbine blades, steam turbines blades, and helicopter rotor blades as well) the problem corresponding to (27) can be solved by using numerical methods. However, this is not surprising, since even in the classical linear theory of prismatic beams analytical solutions are available for a limited number of cases only (see, for example, Love 1944).

As discussed in section 3, we are assuming the smallness of the warpings, strains and local rotations. Moreover, hereafter we choose the current local frames to be tangent to the current center-line and we include possible shear deformations within the warping fields. In addition, ~~here, we assume~~ with the aim of showing a first analytical solution for bi-tapered cross-sections, in this section we neglect the effects of the initial twist ~~is negligible and the initial taper is uniform (A_{B22}=A_{B33}=A_0~~of the cross-sections. Then, we proceed to find a solution that satisfies (27).

~~The 3D nonlinear problem is split, as discussed, into a 2D linear problem and a 1D nonlinear problem. The 1D problem can be solved numerically when the stiffness properties of the cross-sections have been obtained from the 2D problem. Here we focus the attention on this latter problem, leaving to a successive paper the discussion on the numerical procedure we have implemented in MatLab to solve the 1D nonlinear problem. For the 2D problem, in the considered case~~Doing so, the Euler-Lagrange equations corresponding to (27), ~~for the effects of~~ in the case we neglect the terms smaller than ε and maintain the terms related to extension, γ_1, and change of curvatures, k_i, are satisfied by the following warping fields

$$
\begin{aligned}
w_1 &= k_1 \frac{\rho^2 d_3^2 - d_2^2}{\rho^2 d_3^2 + d_2^2} \rho \Lambda^2 z_2 z_3 \\
w_2 &= -\nu\gamma_1 \Lambda z_2 - \nu k_2 \rho \Lambda^2 z_2 z_3 + \nu k_3 \Lambda^2 (\rho^2 z_3^2 - z_2^2)/2 \\
w_3 &= -\nu\gamma_1 \rho \Lambda z_3 + \nu k_3 \rho \Lambda^2 z_2 z_3 - \nu k_2 \Lambda^2 (\rho^2 z_3^2 - z_2^2)/2
\end{aligned}
\tag{28}
$$

where d_2 and d_3 are the semi-major axes of ~~the~~ a reference elliptical cross-section~~.~~ (e.g. the one at 18m from root section in Figure 2). Using this result, we can calculate the corresponding strain and stress fields, (14)-(18), ~~the force and moment~~

stress resultants, (19), and ~~the~~ strain energy function U. For example, if we consider a local frame in the reference cross-section with its origin at the cross-section's center of mass and its axes aligned with the cross-section's principal axes of inertia~~,~~ (as in Figure 2), we can write the 1D strain energy function, U, in the form

$$U = \frac{1}{2} EA\rho\Lambda^2 \gamma_1^2 + \frac{1}{2} GJ_1\rho^2\Lambda^4 k_1^2 + \frac{1}{2} EJ_2\rho^3\Lambda^4 k_2^2 + \frac{1}{2} EJ_3\rho\Lambda^4 k_3^2 \tag{29}$$

In ~~Eq.~~ (29), E is the Young modulus, G is the shear modulus, while A, J_1, J_2 and J_3 are the following geometrical ~~constants~~parameters

$$A = \pi d_2 d_3,\; J_1 = A d_2^2 d_3^2 / (\rho d_3^2 + \rho^{-1} d_2^2),\; J_2 = A d_3^2 / 4,\; J_3 = A d_2^2 / 4 \tag{30}$$

An interesting result is that the initial taper~~, Λ_0,~~ appears explicitly in all the previous relations~~, allowing us to analytically evaluate~~ (in terms of ρ and Λ). In its ~~effect. The~~ turn, this allows an analytical evaluation of its effects ~~of the not uniform initial taper, initial twist and other terms,~~ on the 3D strain ~~energy function, on the strain and stress tensor~~ fields, which can be calculated by using (15) and ~~on~~(28), and which are required to determine the ~~force and moment~~3D stress ~~resultants, will be~~fields (17).

5 Numerical simulations

In this section we provide the results of simulations conducted by using the modeling approach discussed in section 3, which we have implemented into a ~~successive work~~numerical code in MATLAB language. Those results are also compared with the results that can be obtained by 3D-FEM simulations with the commercial software ANSYS.

~~5~~ In particular, we show a first set of test cases in which a beam-like structure with rectangular cross-sections undergoes large displacements, while it is fixed at one end and it is loaded at the other end by a force whose magnitude is progressively increased. In the second set of test cases we consider a more complex geometry, that is, a beam-like structure with elliptical cross-sections, which is curved, twisted and tapered in its reference configuration, while the loading condition is the same as in the first set of test cases. Finally, in the third set of test cases, we consider (four) different beam-like structures under the same loading condition. In particular, we consider a first prismatic structure with elliptical cross-sections. The second structure is a modification of the first one, on which we maintain the same cross-section at 18m from root and we add the taper according to the taper coefficients of Figure 2. Starting from this latter, we consider a third structure which includes the twist of the cross-sections, assuming the twist law of Figure 2. The fourth one is a curved, twisted and tapered structure obtained by the third one (tapered and twisted) by adding the center-line curvature. Then, we compare the results obtained by simulating the behavior of these four structures to shows the effects related to their geometrical differences.

In all the cases, the displacements of the points of the reference center-line are calculated by solving the center-line nonlinear problem through a numerical procedure we have implemented in MATLAB language, which is based on the kinematic, constitutive and balance equations introduced in section 3. In particular, we use the constitutive model introduced in section

3.2 to relate stress resultants and strain measures. We define the local frames orientation by using Euler angles and simulate orientation changes in terms of derivatives of those angles over the arch-length, s (see, for example, Pai 2003). We use the balance equations for the stress resultants introduced in section 3.3. Finally, we integrate (numerically) the resulting set of ordinary differential equations with respect to the arch-length, s. The results of this procedure are illustrated hereafter.

5.1 First set of test cases

In this set of test cases we consider a beam-like structure with rectangular cross-sections undergoing large displacements, while it is clamped at one end (i.e. the root) and it is loaded at the other end (i.e. the tip) by a force, F, whose magnitude is progressively increased (see Figure 3). The center-line length is d_1=90m. The cross-section sizes are d_2=8m (edge-wise) and d_3=2m (flap-wise). The material properties are summarized by reference values of the Young's modulus, 70GPa, and Poisson's ratio, 0.25. The flap-wise tip force, F, varies from 100kN to 75000kN.

The simulations are run for different values of the tip force. The model we have implemented in MATLAB language to solve the non-linear problem provides results on the deformed configuration of the structure (e.g. Figure 3, left) within a few seconds. In all the cases, the simulation time is less than 2.4s. It is significantly less than that required by the corresponding non-linear 3D-FEM simulations carried out on the same computer, while the accuracy of the results is almost the same. A summary of the obtained results, in terms of global displacements and simulation times, is shown in Figures 3 and 4.

In particular, Figure 3 (left) shows the un-deformed shape (for F=0), as well as the deformed shapes for F equal to 10000kN, 25000kN and 50000kN. Figure 3 (right) shows the 3D-FEM deformed shape for F=25000kN (the coloured legend is for the flap-wise displacements). Then, Figure 4 (left) provides a comparison between the tip displacements obtained with the linear 3D-FEM, the nonlinear 3D-FEM and our model (therein referred to as 3D-BLM). It also shows the differences (between the non-linear 3D-FEM and the 3D-BLM) in terms of tip displacements and simulation times for the considered cases.

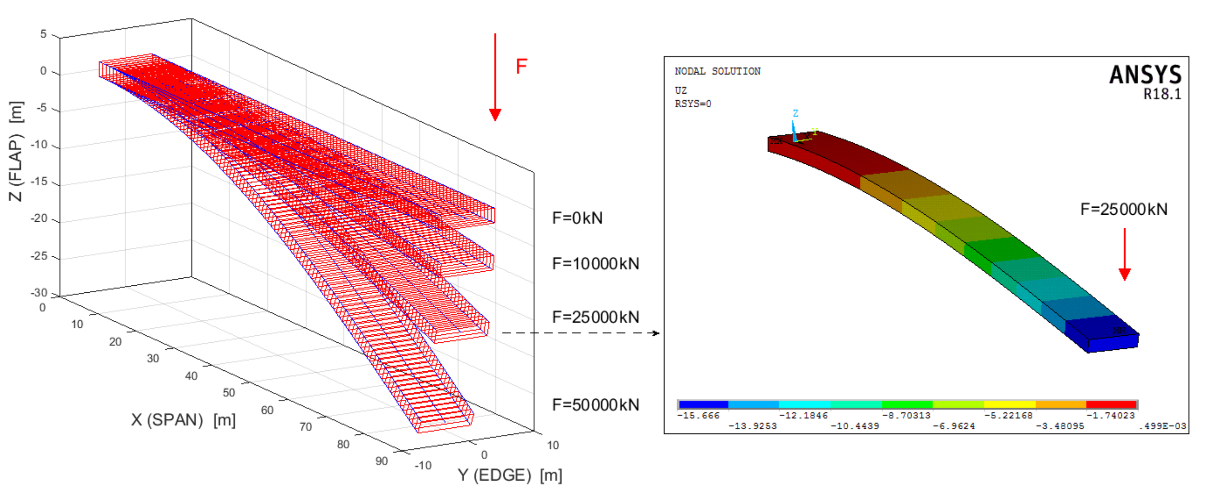

Figure 3: Global deformation with 3D-BLM for F increasing (left) and with 3D-FEM for F=25000kN (right)

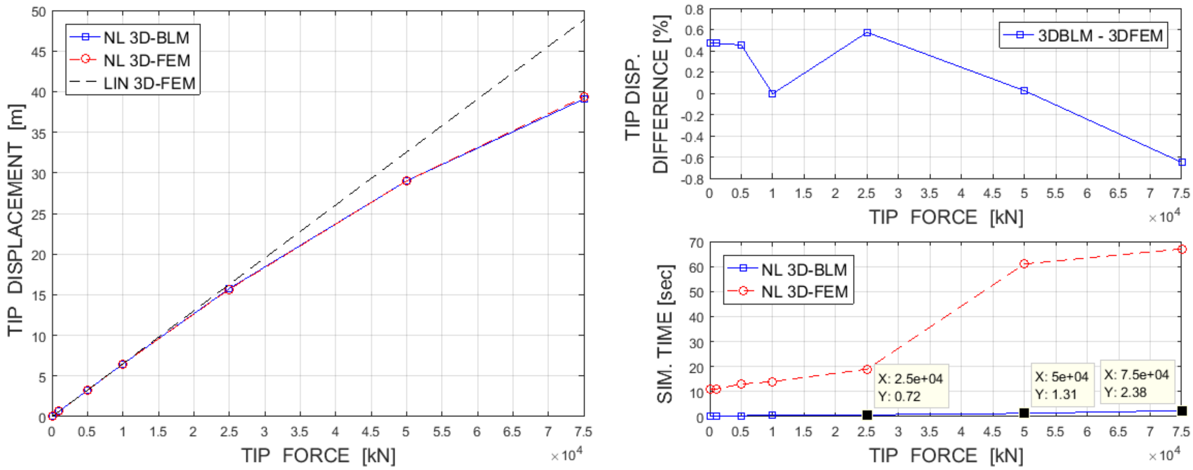

Figure 4: Comparison of tip displacements (left), tip displacements differences and simulation times (right)

5.2 Second set of test cases

Let's now consider a more complex beam-like structure, more precisely, a 90m curved center-line with constant curvatures, which schematizes a pre-bent and swept beam whose tip is moved 4m edgewise and 3m flapwise, as in Figure 2. The local frames in the reference state are characterized by a pre-twist of 20deg/m. The reference cross-section at 18m from root is an ellipse whose semi-major axes are d_2=2m (edge-wise) and d_3=0.5m (flap-wise). The sizes of the other cross-sections change according to the taper coefficients of Figure 2. For what the material properties are concerned, they are summarized by reference values of the Young's modulus, 70GPa, and Poisson's ratio, 0.25. Finally, the structure is clamped at its root and it is loaded by a flap-wise tip force, F, which varies from 100kN to 1000kN.

The simulations are run for different values of the tip force. The model we have implemented in MATLAB language to solve the non-linear problem provides the results about the deformed configurations (see, for example, Figure 5, left) within less than 2.7 seconds. As in the first set of test cases, the simulation time is significantly less than that required by the nonlinear 3D-FEM simulations (they differ by at least one or two order of magnitude), while the accuracy of the results is again almost the same. Hereafter, we continue by showing some other information our model can provide. In particular, we can obtain the displacement fields of the points of the reference center-line (Figure 6), as well as the change of curvatures of the beam-like structure (Figure 7, left) and the corresponding moment stress resultant (Figure 7, right). The moment components are in the current local frame, a_i, whose orientation has been determined as part of the solution of the nonlinear problem. For example, the orientation of the current local frame, a_i, can be provided in terms of a set of Euler angles. See Figure 8. In this case we have considered the set of Euler angles corresponding to a first rotation, θ, about the initial z-axis, a second rotation, γ, about the intermediate y-axis, and a third rotation, ψ, about the final x-axis.

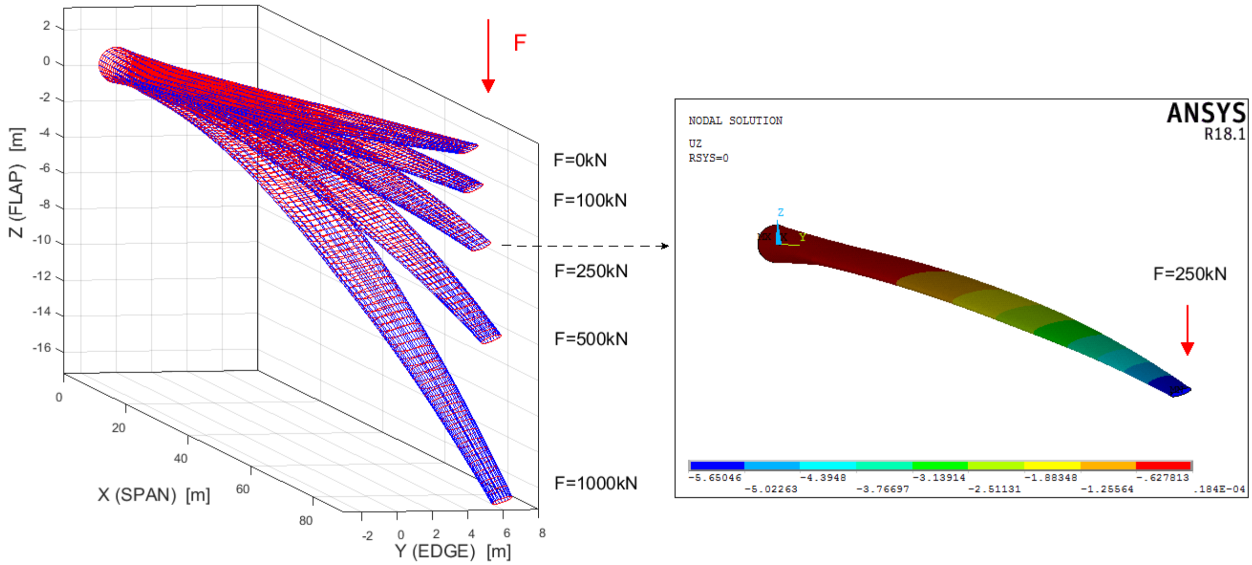

Figure 5: Global deformation with 3D-BLM for F increasing (left) and with 3D-FEM for F=250kN (right)

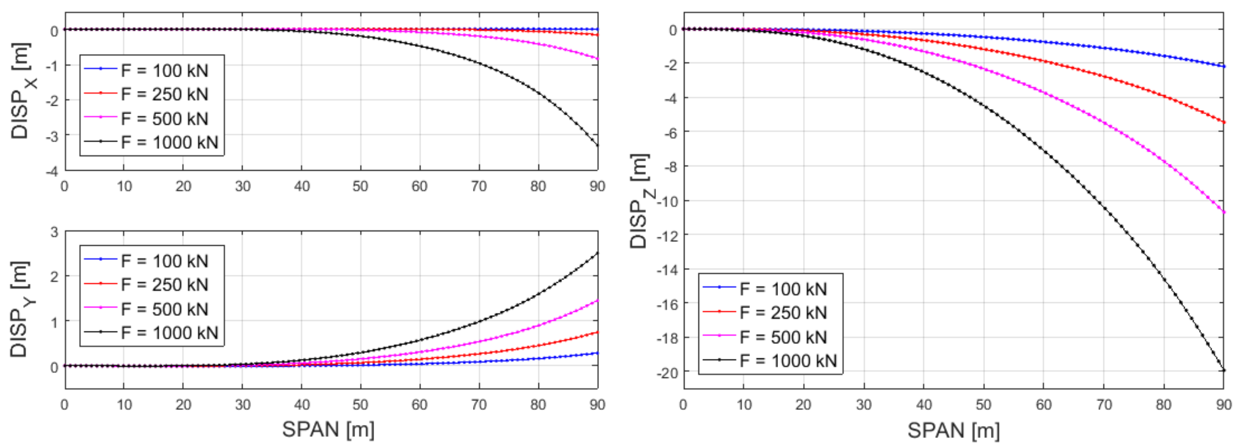

Figure 6: Displacement of the points of the reference center-line with 3D-BLM for F increasing

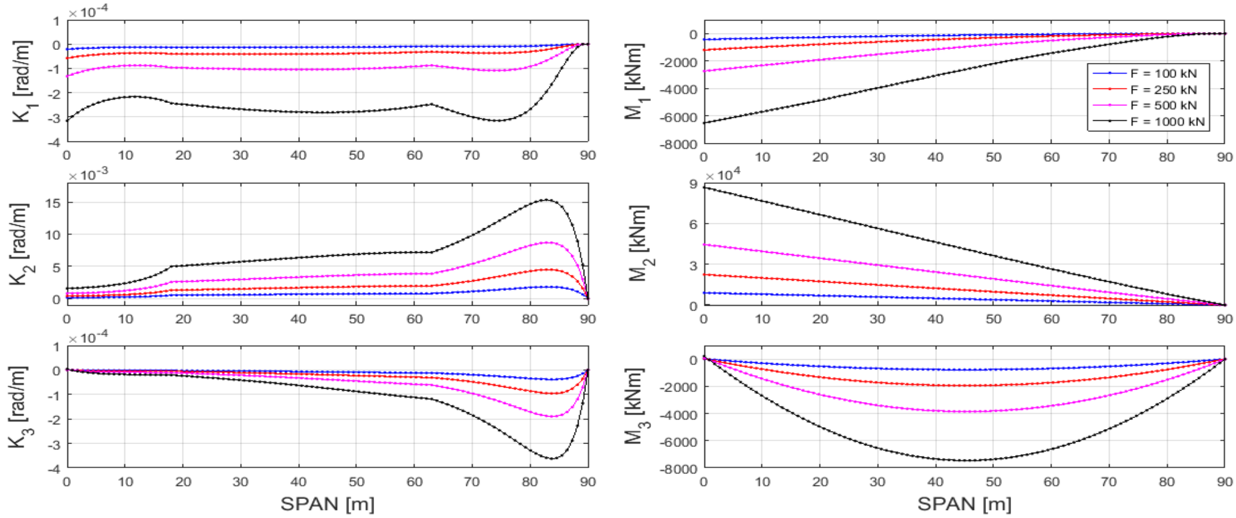

Figure 7: Changes of curvatures (left) and moment stress resultants (right) with 3D-BLM for F increasing

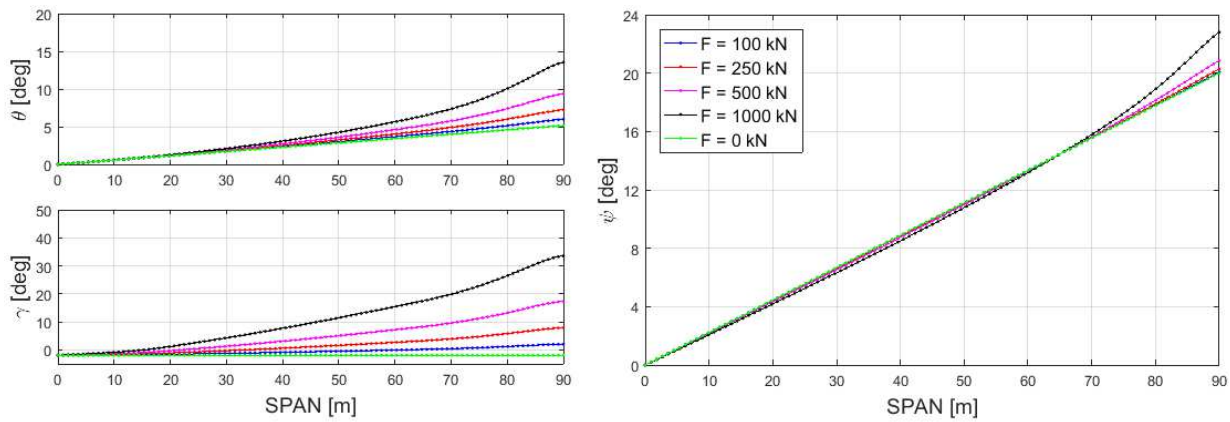

Figure 8: Local frames orientation in terms of Euler angles before (green-lines) and after deformation

5.3 Third set of test cases

Here we consider different beam-like structures, starting with a prismatic elliptical one, including step by step the taper and twist of the cross-sections and, finally, the curvature of the center-line, as discussed in the beginning of section 5. Note that the "curved-twisted-tapered" case considered here coincides with that discussed with more details in section 5.2 (see Figures 5-8, F=250kN). We proceed by simulating the bahavior of these four structures under a flap-wise tip force of 250kN. Then, we analyze the obtained results to show the effect of their geometric differences on their mechanical behaviour. A summary of the obtained results is hereafter. In particular, Figure 8 shows the reference and deformed states of the prismatic structure (left) and the deformed states of the non-prismatic ones (right), while Figure 9 shows the displacements of their center-lines points. The main effect of the considered tip force is a displacement in the z-direction, in all the cases, with a displacement in the y-direction that we have only for the cases "tapered-twisted" and "tapered-twisted-curved", as it is expected.

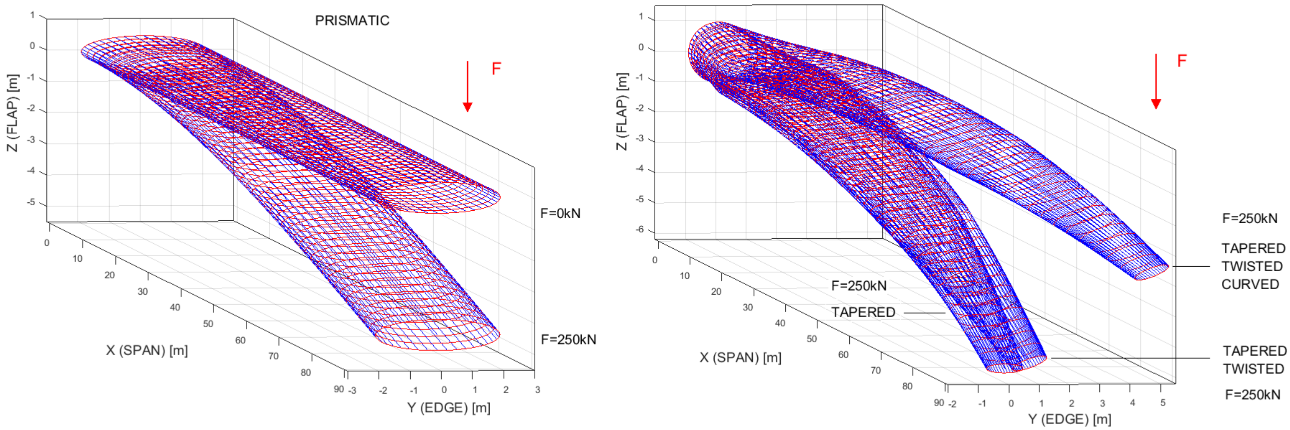

Figure 9: Prismatic structure before and after deformation (left) and non-prismatic structures after deformation (right)

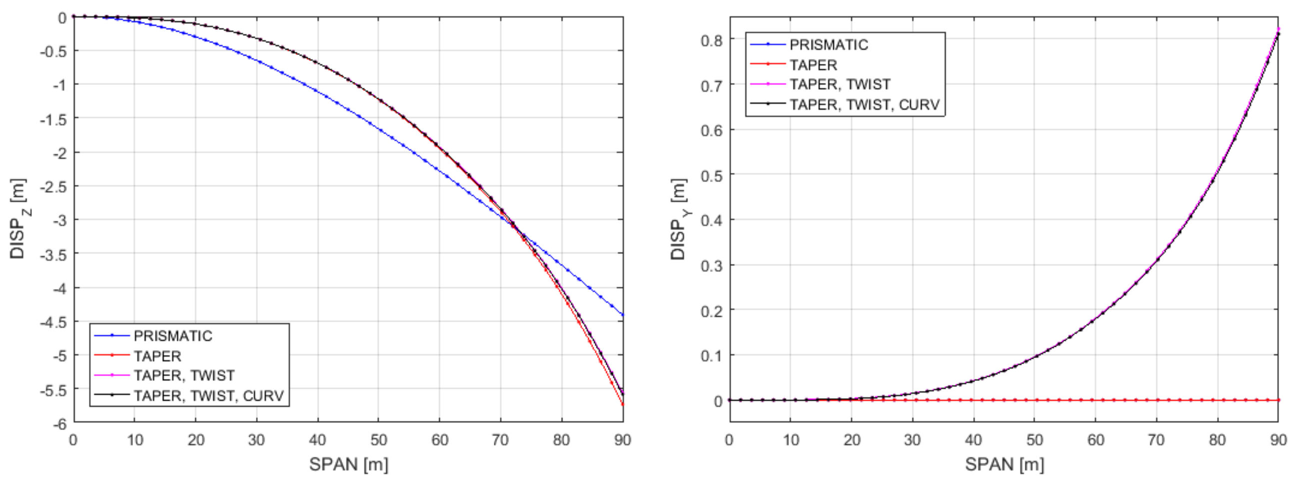

Figure 10: Displacement of the center-line points of the prismatic structure (blue) and non-prismatic structures (other colours)

The obtained results have been compared with those of the 3D-FEM commercial software, for this third set of test cases too, confirming the computational efficiency and accuracy of the previous sets of test cases. A summary of those results is shown in Figure 10, which provides a comparison in terms of tip displacements (components) for the four cases considered here.

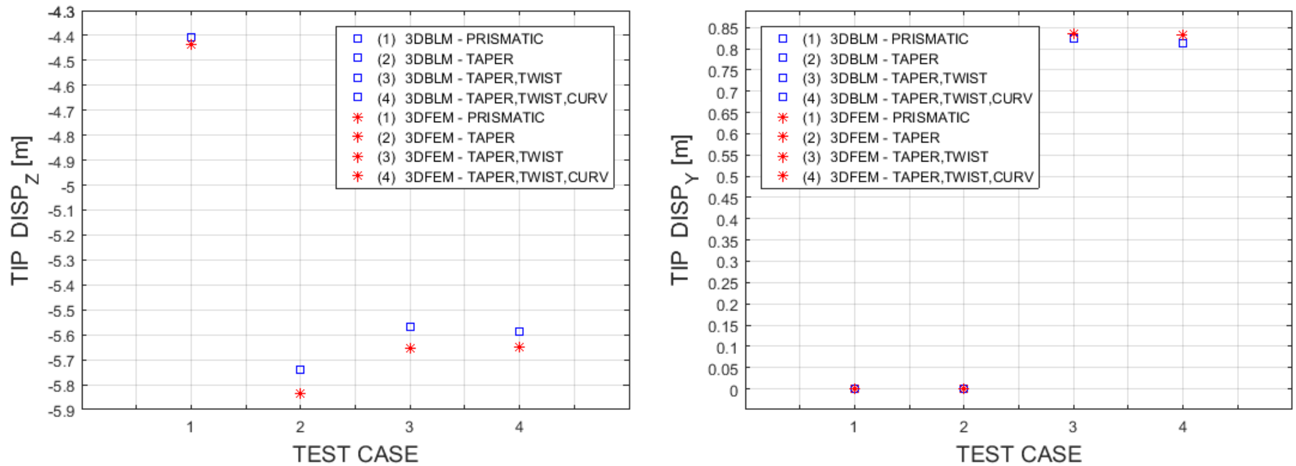

Figure 11: Tip displacements with 3D-BLM (blue) and 3D-FEM (red) for different beam-like structures at F=250kN

As verified by many simulations and shown in the examples, the proposed approach performs well in terms of computational efficiency and accuracy. It can be used to study the mechanical behavior of beam-like structures, which are curved, twisted and tapered in their reference unstressed state and undergo large global displacements. It can provide information on the deformed configurations of those structures, such as their displacement fields, as well as the corresponding strain and stress measures. It is worth noting that it is suitable for beam-like structures with generic reference cross-sections shapes. However, as already pointed out, for bi-tapered elliptical cross-sections we have analytical solutions in terms of warping fields, while for generic reference cross-sections shapes the problem (27) has to be solved by using numerical methods.

6 Conclusions

~~Modern~~Wind turbine blades, as well as helicopter rotor blades, steam turbine blades and many other engineering structures, can be considered complex (non-prismatic) beam-like structures, with one dimension much larger than the other two and a shape that is curved, twisted and also tapered already in the reference unstressed state. Their mechanical behavior can be simulated through suitable 3D beam models, which are computationally efficient, accurate and explicitly consider ~~their~~the main geometrical ~~characteristics, possible~~design features of those structures, the large ~~displacements~~deflections of their center-line, and ~~3D local~~the in- and out-of-plane warping of their cross-sections. In this work, curved, twisted and tapered beam-like structures have been modeled analytically. ~~The~~Their main geometrical ~~design features, such as the curvatures of the center-line and the twist and taper of the cross-sections,~~characteristics have been explicitly included in the model. The warping displacement has been thought of as an additional small motion superimposed to the global ~~generic~~ motion of the ~~cross-sectional~~local frames. The resulting model is suitable to simulate large deflections of the center-line, large rotations of the ~~cross-sectional~~local frames and small deformation of the cross-sections. The strain tensor has been calculated analytically in terms of the geometrical parameters of the ~~structure~~considered structures, the 1D strain measures and the 3D warping fields. ~~The same has been done for the 3D and 1D energy functions.~~ An approach based on an energy ~~functionals~~functional and ~~the slenderness of the structure~~a variational statement have been used to obtain suitable warping fields. The principle of expended power for curved, twisted and tapered beam-like ~~structure~~structures has been discussed, ~~as well as~~along with the balance equations for the force and moment stress resultants. Finally, ~~an application example, which includes~~ analytical results~~, has~~ and numerical examples, which include comparison with 3D FEM simulations, have been presented to show the ~~information~~ effectiveness of the proposed modeling approach and the information it can provide.

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