Optimum induction distribution for fixed bending moment

Reference values:

$$\rho := 1.225 \qquad U_0 := 10 \qquad R_0 := 50 \qquad a := \frac{1}{3} \qquad B := 3 \qquad \lambda := 8$$

$$F(x) := \frac{2}{\pi} \cdot \text{acos}\left[e^{-\frac{(1-x)\cdot B \cdot \lambda}{2\cdot(1-a)}}\right]$$

Prandtl tip loss factor - to switch off make B or λ large

Reference case 1: Standard optimisation for maximum power

$$Cp := \int_0^1 8\cdot a \cdot (1-a)^2 \cdot x \cdot F(x) \, dx = 0.549$$

$$Ct := \int_0^1 8\cdot a \cdot (1-a) \cdot x \cdot F(x) \, dx = 0.824$$

$$Cm_r := \int_0^1 8\cdot a \cdot (1-a) \cdot x^2 \cdot F(x) \, dx = 0.531$$

$$M_0 := 0.5\cdot \rho \cdot U_0^2 \cdot \pi \cdot R_0^3 \cdot Cm_r = 1.277 \times 10^7 \qquad \text{Reference total bending moment}$$

$$P_0 := 0.5\cdot \rho \cdot U_0^3 \cdot \pi \cdot R_0^2 \cdot Cp = 2.643 \times 10^6 \qquad \text{Reference rotor power}$$

$$T_0 := 0.5\cdot \rho \cdot U_0^2 \cdot \pi \cdot R_0^2 \cdot Ct = 3.964 \times 10^5 \qquad \text{Reference rotor thrust}$$

Reference case 2: Optimum axial induction to maximise power for variable radius and fixed total moment   :

$$a := 0.2 \qquad Ct(a) := 4\cdot a \cdot (1-a) \qquad Ct(a) = 0.640 \qquad Cp(a) := \int_0^1 8\cdot a \cdot (1-a)^2 \cdot x \cdot F(x) \, dx$$

$$Cp(a) = 0.475$$

$$Cm(a) := \int_0^1 8\cdot a \cdot (1-a) \cdot x^2 \cdot F(x) \, dx \qquad Cm(a) = 0.382$$

$$R_{opt} := \left[\frac{M_0}{\left(0.5\cdot \rho \cdot U_0^2 \cdot \pi \cdot Cm(a)\right)}\right]^{\frac{1}{3}} = 55.786 \qquad M_0 = 1.277 \times 10^7$$

$$\frac{R_{opt}}{R_0} = 1.116$$

11% increase in radius

$$P := 0.5 \cdot \rho \cdot U_0^3 \cdot \pi \cdot R_{opt}^2 \cdot Cp(a) = 2.842235 \times 10^6$$

$$\frac{P}{P_0} = 1.076$$

7.6% increase in power

Exactly same results with tip loss on or off as F cancels in M_0 and Cm

Axial induction function:

$$N := 10 \qquad a := \frac{1}{3} \qquad p := 1$$

Arbitrary starting values for the optimisation except that the solutions for maximisation with small and large N initial values appear to be different.

$$a1(x, N, p) := a \cdot \left(1 - x^N\right)^p$$

A definition of axial induction with 3 DOF (a,N,p) that will tend to zero at x=1 and be positive over [0,1].

$a1(x, 0.5, 1)$
———

$a1(x, 0.5, 0.5)$
- - - -

$a1(x, 1.5, p)$
— · —

$a1(x, 100, 1)$
— · · —
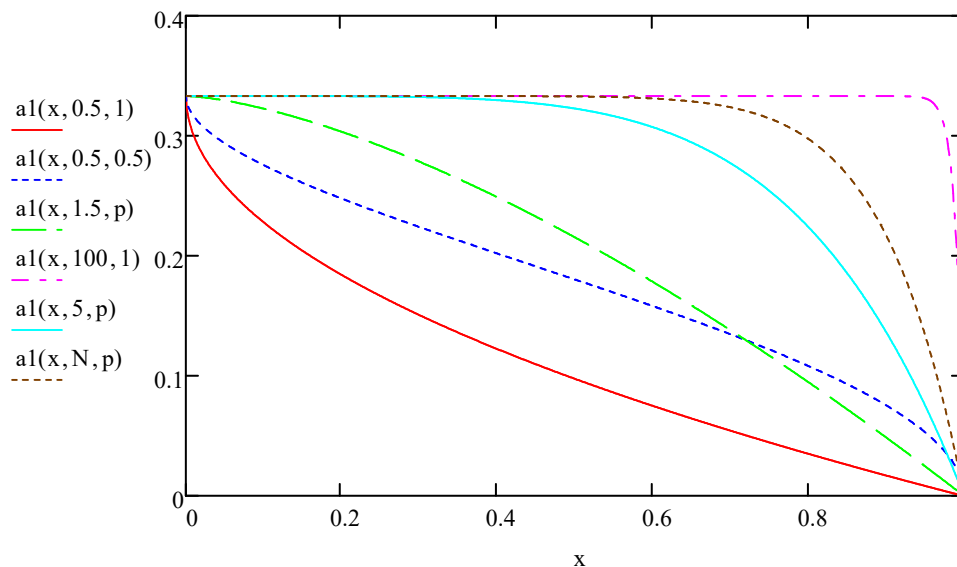
$a1(x, 5, p)$
———

$a1(x, N, p)$
- - - -

Fig 1   Illustration of variety of possible axial induction distributions

$$P(N, a, p) := \frac{4 \cdot \rho \cdot \pi \cdot U_0{}^3 \cdot M_0{}^{\frac{2}{3}} \cdot \displaystyle\int_0^1 a \cdot \left(1 - x^N\right)^p \cdot \left[1 - a \cdot \left(1 - x^N\right)^p\right]^2 \cdot x \cdot F(x)\, dx}{\left[4 \cdot \rho \cdot U_0{}^2 \cdot \pi \cdot \displaystyle\int_0^1 a \cdot \left(1 - x^N\right)^p \cdot \left[1 - a \cdot \left(1 - x^N\right)^p\right] \cdot x^2 \cdot F(x)\, dx\right]^{\frac{2}{3}}}$$

$$R_{opt}(N, a, p) := \left[\frac{M_0}{\left[4 \cdot \rho \cdot U_0{}^2 \cdot \pi \cdot \displaystyle\int_0^1 a \cdot \left(1 - x^N\right)^p \cdot \left[1 - a \cdot \left(1 - x^N\right)^p\right] \cdot x^2 \cdot F(x)\, dx\right]}\right]^{\frac{1}{3}}$$

Given

$$a \le \frac{1}{3}$$

$$\text{OPT} := \text{Maximize}(P, N, a, p) \qquad \text{OPT} = \begin{pmatrix} 1.504 \\ 0.331 \\ 1.125 \end{pmatrix} \qquad a := \text{OPT}_1 \qquad a = 0.331$$

$$N := \text{OPT}_0 \qquad N = 1.504$$

$$p := \text{OPT}_2 \qquad p = 1.125$$
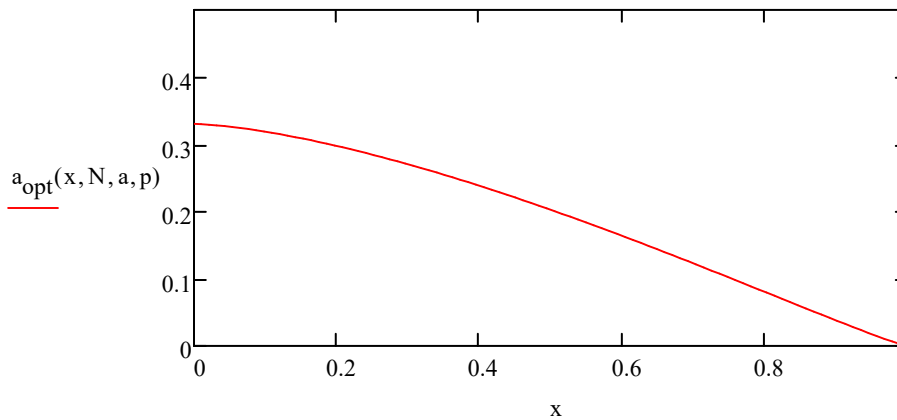
$$a_{opt}(x, N, a, p) := a \cdot \left(1 - x^N\right)^p$$

Fig2　The optimum axial induction distribution for maximum power at fixed M_0

$$R_{opt}(N,a,p) = 67.133 \qquad \frac{R_{opt}(N,a,p)}{R_0} = 1.343$$

$$P(N,a,p) = 2.957 \times 10^6 \qquad \frac{P(N,a,p)}{P_0} = 1.119$$

Optimum distribution of axial induction for constrained radius expansion

Given

$$a \le \frac{1}{3}$$

$$R_{opt}(N,a,p) \le 1.067 \cdot R_0$$

$$OPT := Maximize(P,N,a,p) \qquad OPT = \begin{pmatrix} 0.417 \\ 0.333 \\ 0.136 \end{pmatrix} \qquad \begin{array}{ll} a := OPT_1 & a = 0.333 \\ \\ N := OPT_0 & N = 0.417 \end{array}$$

$$ax(x,N,a,p) := a_{opt}(x,N,a,p) \qquad\qquad p := OPT_2 \qquad p = 0.136$$
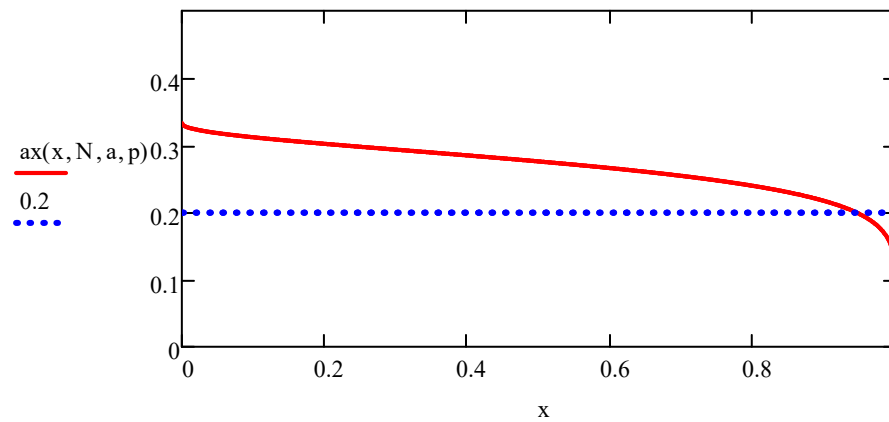

Fig3   The optimum axial induction distribution for maximum power at fixed M_0 with constrained radius expansion

$$R_{opt}(N,a,p) = 53.351 \qquad \frac{R_{opt}(N,a,p)}{R_0} = 1.067 \qquad \begin{array}{l} a = 0.333 \\ \\ N = 0.417 \end{array}$$

$$P(N,a,p) = 2.843 \times 10^6 \qquad \frac{P(N,a,p)}{P_0} = 1.076 \qquad p = 0.136$$

Check that constraint on M_0 is met

$$M_0 := 0.5 \cdot \rho \cdot U_0^2 \cdot \pi \cdot R_0^3 \cdot Cm_r = 1.277 \times 10^7$$

$$M_{check} := 0.5 \cdot \rho \cdot U_0^2 \cdot \pi \cdot R_{opt}(N,a,p)^3 \cdot \int_0^1 8 \cdot a \cdot \left(1 - x^N\right)^p \cdot \left[1 - a \cdot \left(1 - x^N\right)^p\right] \cdot x^2 \cdot F(x) \, dx = 1.277 \times 10^7$$
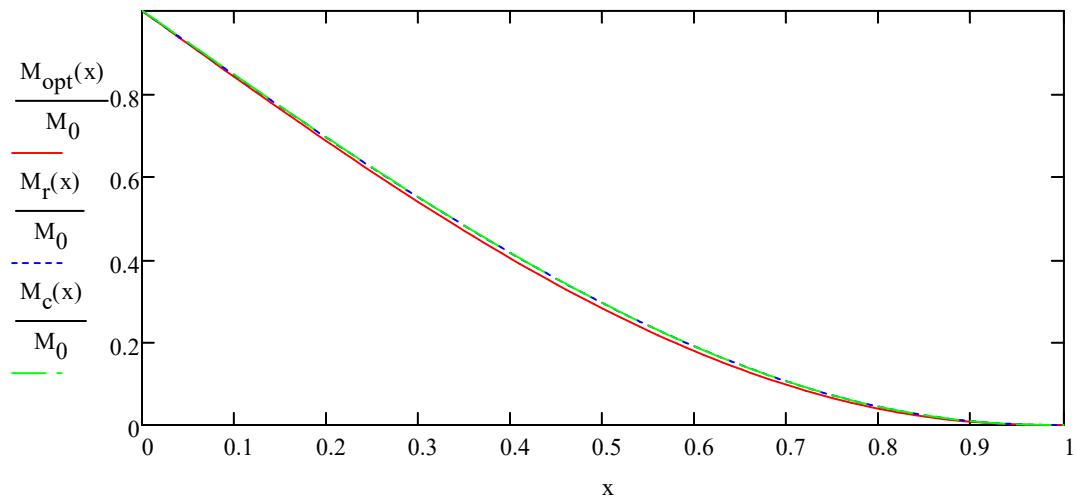
Bending moment distributions

$$M_{opt}(x) := 0.5 \cdot \rho \cdot U_0^2 \cdot \pi \cdot R_{opt}(N,a,p)^3 \cdot \int_x^1 8 \cdot a \cdot \left(1 - y^N\right)^p \cdot \left[1 - a \cdot \left(1 - y^N\right)^p\right] \cdot (y - x) \cdot y \cdot F(y) \, dy$$
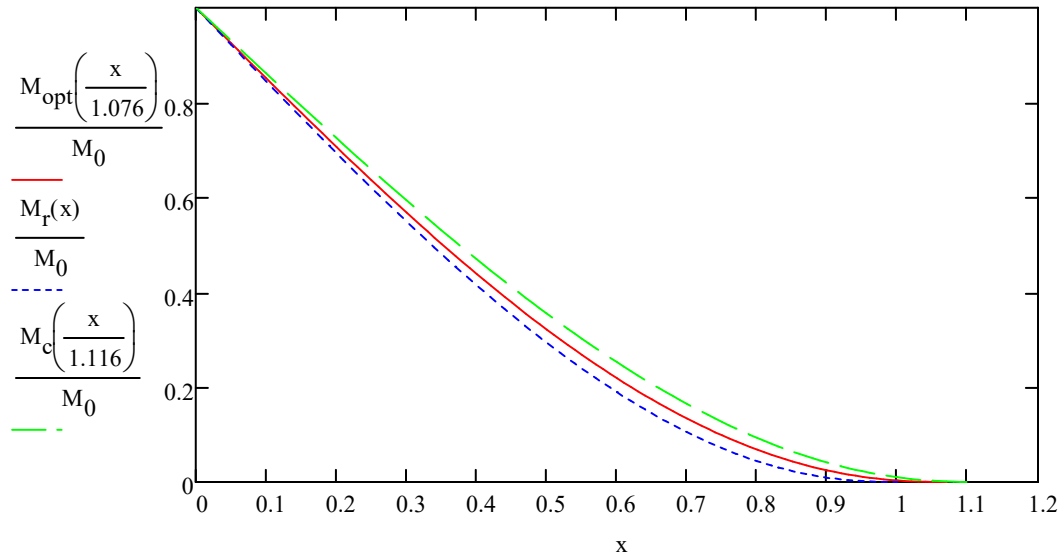
$$a := \frac{1}{3} \qquad b := \frac{1}{5}$$

$$M_r(x) := 0.5 \cdot \rho \cdot U_0^2 \cdot \pi \cdot R_0^3 \cdot 8 \cdot a \cdot (1 - a) \cdot \int_x^1 (y - x) \cdot F(y) \cdot y \, dy$$

$$M_c(x) := 0.5 \cdot \rho \cdot U_0^2 \cdot \pi \cdot R_0^3 \cdot 1.116^3 \cdot 8 \cdot b \cdot (1 - b) \cdot \int_x^1 (y - x) \cdot F(y) \cdot y \, dy$$

$$x := 0, 0.01 .. 1.12$$

The legend on the left of the plot:

$$\dfrac{M_{opt}\!\left(\dfrac{x}{1.076}\right)}{M_0}$$

(red solid line)

$$\dfrac{M_r(x)}{M_0}$$

(blue dashed line)

$$\dfrac{M_c\!\left(\dfrac{x}{1.116}\right)}{M_0}$$

(green dash-dot line)

$$\dfrac{M_{opt}\!\left(\dfrac{0.5}{1.076}\right)}{M_r(0.5)} = 1.095$$

Axial thrust force

$$Ct_{opt} := \int_0^1 8\cdot a\cdot\left(1 - x^N\right)^p\cdot\left[1 - a\cdot\left(1 - x^N\right)^p\right]\cdot x\cdot F(x)\ dx = 0.698$$

$$T_{opt} := 0.5\cdot\rho\cdot U_0{}^2\cdot\pi\cdot R_{opt}(N,a,p)^2\cdot Ct_{opt} = 3.825 \times 10^5$$
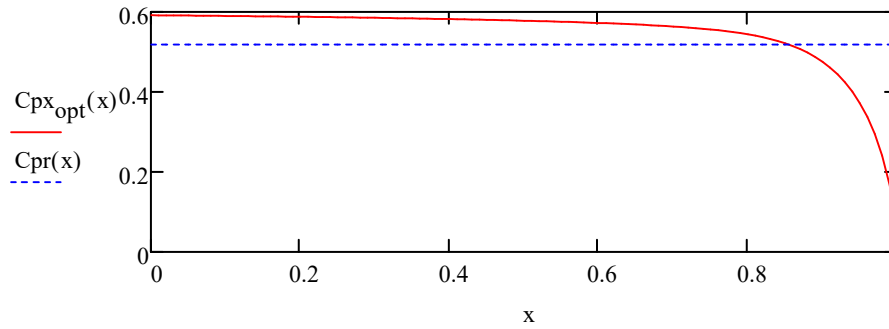
$$\dfrac{T_{opt}}{T_0} = 0.965$$

Thrust reduction is $\left(1 - \dfrac{T_{opt}}{T_0}\right)\cdot 100 = 3.497$ %

Power coefficient

$$Cp_{opt} := \int_{0}^{1} 8 \cdot a \cdot \left(1 - x^N\right)^p \cdot \left[1 - a \cdot \left(1 - x^N\right)^p\right]^2 \cdot x \cdot F(x) \, dx = 0.519$$

$$Cpx_{opt}(x) := 4 \cdot a \cdot \left(1 - x^N\right)^p \cdot \left[1 - a \cdot \left(1 - x^N\right)^p\right]^2 \cdot F(x) \qquad Cpr(x) := Cp_{opt}$$

radial distribution of Cp

radial distribution of axial induction