# An impulse-based derivation of the Kutta-Joukowsky equation for wind turbine thrust

Eric J. Limacher[1] and David H. Wood[2]

[1]Department of Mechanical Engineering, Federal University of Pará, Belém, Brazil
[2]Department of Mechanical and Manufacturing Engineering, University of Calgary, Calgary T2N 1N4, AB, Canada.

**Correspondence:** David H. Wood (dhwood@ucalgary.ca)

**Abstract.** Using the concept of impulse in control volume analysis, we derive general expressions for wind turbine thrust in a constant, spatially uniform wind. The absence of pressure in the impulse equations allows for their application in the near wake, where the flow is assumed to be steady in the frame of reference rotating with the blades. The assumption of circumferential uniformity in the near wake — as applies when the number of blades or the tip speed ratio tends to infinity — is needed to reduce these general expressions to the Kutta-Joukowsky (KJ) equation for blade-element thrust. The present derivation improves upon the classical derivation based on the Bernoulli equation by allowing the flow to be rotational in the near wake. The present derivation also yields intermediate expressions for thrust that allow for a finite number of blades and trailing vortex sheets of finite thickness. For the circumferentially uniform case, our analysis suggests that the magnitudes of the radial velocity and the axial induction factor must be equal somewhere on the plane containing the rotor, and we cite previous studies that show this to occur near the rotor tip across a wide range of thrust coefficients. The derivation reveals one further complication; when deriving the KJ equations using annular control volumes, the existence of vorticity on the lateral control surfaces may cause the local blade loading to differ from the KJ equation, but the magnitude of these deviations is not explored. This complication is not visible to the classical derivation due to its neglect of vorticity.

## 1 Introduction

Blade element theory (BET) for wind turbines uses the fundamental assumption that the forces acting on the elements comprising the rotor blades are given by the Kutta-Joukowsky (KJ) theorem. The thrust and torque are balanced by the change in the axial and angular momentum, respectively, of the flow through a control volume (CV) enclosing the rotor; the combination of BET and momentum theory gives rise to blade-element momentum (BEM) theory. BEM is developed in many texts on wind turbine aerodynamics, such as Burton *et al.* (2011), Wood (2011), Hansen (2015), Schmitz (2019), and Schaffarczyk (2020), so it is unnecessary to repeat it here. Herein, the KJ equation for blade-element thrust will be rederived using impulse theory, and the conditions of its validity are examined.

Although the KJ equations are generally introduced as assumptions in BEM theory, they can be derived using the unsteady Bernoulli equation. This was shown as early as Glauert (1935), and can also be found in Sørensen (2016) and van Kuik (2018).

Assuming the trailing wake to be infinitely thin to permit the use of the Bernoulli equation, and representing the rotor's power extraction by a sudden drop in pressure across the rotor plane, sectional thrust can be expressed as

$$\frac{1}{\rho}\frac{dT}{dx} = \int_{0}^{2\pi} \left(\frac{1}{2}u_\theta^2 - \lambda u_\theta x\right) x d\theta, \tag{1}$$

where $\rho$ is fluid density, $T$ is the rotor thrust, $x$ is the radial coordinate, $\lambda$ is the tip speed ratio, and $\theta$ is the azimuthal angle. $u_\theta$ is the circumferential velocity just behind the rotor. For ease of reference, we have included our own version of the classical Bernoulli-based derivation of Equation (1) in Appendix A.

In keeping with typical notation in BEM analysis, we define $u_\theta = -2w$ on this near-wake plane, such that positive $w$ corresponds to positive torque. This substitution gives

$$\frac{1}{\rho}\frac{dT}{dx} = \int_{0}^{2\pi} \left(2w^2 + 2\lambda w x\right) x d\theta. \tag{2}$$

$w$ is generally interpreted as the circumferential induced velocity at the rotor plane itself, which is assumed in BEM theory to be half that at the plane just behind the rotor. All quantities in Equations (1) and (2) have been normalized by freestream velocity and rotor radius.

When the flow is circumferentially uniform, Equation (1) reduces to the KJ lift theorem, as will be shown later using Stokes' theorem. Here, we merely wish to point out the contradiction in this classical approach. Although the derivation of (1) required us to assume irrotational flow everywhere in the near wake — save in the vortical wake of vanishing thickness — the assumption of circumferental uniformity will lead to non-zero vorticity on the near-wake plane except in unrealistic special cases (i.e. $u_\theta \equiv 0$ or $u_\theta = k/x$, where $k$ is a constant). Using impulse theory, we offer an improved derivation of the KJ equation for blade-element thrust that avoids this contradiction. The derivation also offers new insights regarding the relative magnitude of radial velocity in the expanding flow upwind and through the rotor.

The remainder of the paper is organized as follows. Since most of the wind energy research community may be unfamiliar with the concept of "vortical impulse" or "hydrodynamic impulse" — usually referred to simply as "impulse" — we offer a brief introduction in section 2. Section 3 then manipulates the general CV equations furnished by impulse theory to yield simplified integral expressions of wind turbine thrust. The reader interested primarily in the consequences of the present impulse analysis for BEM may skip ahead to section 4. Finally, our conclusions are presented in section 5.

## 2 Background on Impulse Theory

Impulse theory expresses fluid-dynamic forces in terms of the first moment of vorticity, e.g. Wu et al. (2015):

$$\frac{1}{N-1}\int_{V} \mathbf{x} \times \mathbf{\Omega} dV, \tag{3}$$

where $V$ is some fluid volume, $N$ is the dimension of the space, $\mathbf{x}$ is the position vector, and $\mathbf{\Omega}$ is the vorticity vector. The use of the word "impulse" may confuse the unfamiliar reader; it is an imprecise yet well-established nomenclature. Lighthill (1986) showed that the first moment of vorticity is equal to the impulse (i.e. the integral of force applied over time) necessary to establish an unbounded vortical flow from rest (the domain is unbounded, but the region of vortical fluid remains bounded). Since then, the term "impulse" has been co-opted to refer specifically to the first moment of vorticity, as recounted in section 2.5 of Limacher (2019), and we will continue to use this terminology here.

The introduction of the concept of impulse removes the pressure and introduces vorticity to the equations of linear momentum conservation. This approach can be traced at least as far back as Thomson (1882), who sought to determine the speed of a vortex ring. In their review, Wu et al. (2015) recount that exact impulse-based expressions for aerodynamic force were derived independently by Burgers (1921), Wu (1981) and Lighthill (1986). Discussions of impulse formulations can also be found in Lamb (1932), Batchelor (1967), and Saffman (1992). They have recently gained popularity for use with planar or volume meaurements of fluid velocities from particle image velocimetry and related techniques, e.g. Rival & van Oudheusden (2017), Limacher et al. (2019a), and Limacher et al. (2020).

Derivation of an impulse-based force formulation begins with the conventional application of the Reynolds transport theorem to the momentum equation, and proceeds to manipulate the equations by means of vector calculus identities. For a full presentation, we recommend Noca's (1997) doctoral thesis, as only a brief overview is given here.

For an inertial CV in an incompressible fluid, the equation for force, $\mathbf{F}$, is given in many fluid mechanics texts and by Noca's (1997) Equation (3.36):

$$\frac{\mathbf{F}}{\rho} = -\frac{d}{dt}\int_V \mathbf{U} dV + \oint_S \mathbf{n}\cdot\left(-\frac{P}{\rho}\mathbf{I} - \mathbf{U}\mathbf{U} + \mathbf{T}\right) dS \tag{4}$$

where $t$ is time, and $\mathbf{U}$ is the velocity vector, $S$ is the control surface bounding the control volume, $V$, $\mathbf{n}$ is the outward-facing normal on $S$, $\mathbf{I}$ is the unit tensor, $P$ is the pressure and $\mathbf{T}$ is the viscous stress tensor. The $\mathbf{n}\cdot\mathbf{U}\mathbf{U}$ term gives the conventional momentum deficit when the equation is used to determine thrust.

The momentum integral — the first integral on the right-hand side of (4) — can be related to the impulse as given in (3) by the so-called "impulse-momentum identity" (Noca's Equation (3.1)) at the expense of additional terms. By then removing $P$ using the Navier-Stokes equations and other vector identities, Noca obtained his Equations (3.55) and (3.56), which we combine as

$$\frac{\mathbf{F}}{\rho} = -\frac{1}{N-1}\frac{d}{dt}\int_V \mathbf{x}\times\mathbf{\Omega} dV + \oint_S \mathbf{n}\cdot\left(\frac{1}{2}U^2\mathbf{I} - \mathbf{U}\mathbf{U} - \frac{1}{N-1}\mathbf{U}(\mathbf{x}\times\mathbf{\Omega}) + \frac{1}{N-1}\mathbf{\Omega}(\mathbf{x}\times\mathbf{U})\right) dS + \mathbf{F}_v \tag{5}$$

where $N=3$ for our analysis, $\frac{1}{2}U^2 = \frac{1}{2}\mathbf{U}\cdot\mathbf{U}$ is the kinetic energy per unit mass, and $\mathbf{F}_v$ is shorthand for viscous terms evaluated on $S$. In common with other CV analyses of wind turbines, $\mathbf{F}_v$ will be ignored. Equation (5), variations of which can be found in Noca (1997), Wu et al. (2015), and Kang et al. (2017), is the complete, general form of the impulse formulation for fluid-dynamic force. It is as exact as the more familiar Equation (4), and it serves as the starting point for our analysis.

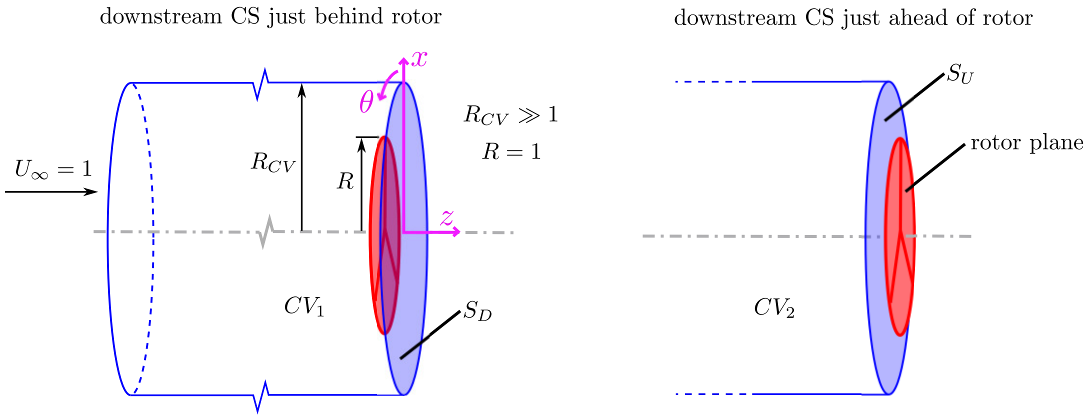


**Figure 1.** Control volumes (CVs) for the present analysis. In both variants, the upstream face is well upstream of the rotor, where the velocity is equal to the freestream, and $R_{CV} \gg R$. The downstream control surface is just downstream and just upstream of the rotor plane in $CV_1$ and $CV_2$, respectively, and the corresponding donwstream control surfaces (CS) are labelled $S_D$ and $S_U$.

## 3 The Impulse Equation for Rotor Thrust

We consider a wind turbine rotating steadily at a tip speed ratio, $\lambda$, defined as the ratio of the circumferential velocity of the blade tips to the wind speed. The latter velocity will be used to normalize all velocities. All lengths are normalized by rotor radius, $R$. We start by using two variants of a cylindrical CV, both of whose upwind face is well ahead of the blades, and whose radius $R_{CV} \gg R$. This ensures that the streamwise velocity along the surface at $R_{CV}$ is equal to the wind speed. The downstream control surface is either just downstream or just upstream of the rotor plane; these variants are called $CV_1$ and $CV_2$, respectively, as shown in figure 1. In both cases, the origin of the cylindrical coordinate system $(x, \theta, z)$ is centred on the downstream face of the CV, as shown on the left of figure 1. $x$ is the radial coordinate, $z$ is the axial coordinate, and $\theta$ is the azimuthal angular coordinate (and the subscript $\theta$ refers to the circumferential direction).

We now consider the contributions of each term in Equation (5) to the thrust, defined as

$$T = \mathbf{F} \cdot \mathbf{e_z}. \tag{6}$$

where $\mathbf{e_z}$ is the unit vector in the $z-$direction, The time derivative of the impulse integral — the first integral in (5) — does not necessarily vanish identically, but its thrust contribution does. Since the vortical wake is assumed to rotate with the blades, only the direction (but not the magnitude) of the impulse integral can change, and its time derivative reduces to

$$\frac{d}{dt} \int_{CV_1} \mathbf{x} \times \mathbf{\Omega} dV = \mathbf{\Lambda} \times \int_{CV_1} \mathbf{x} \times \mathbf{\Omega} dV, \tag{7}$$

where $\mathbf{\Lambda}$ is the rotation vector. For steady rotation, $\mathbf{\Lambda} = \lambda \mathbf{e_z}$, and the thrust contribution of Equation (7) vanishes since $\mathbf{\Lambda}$ and $\mathbf{e_z}$ are parallel.

The first two terms in the surface integral on the right-hand side of Equation (5) can be referred to jointly as the velocity terms. The axial component of the $\mathbf{n}\cdot\mathbf{UU}$ term gives the conventional momentum-deficit term, derived in any textbook on wind turbine aerodynamics:

$$-\mathbf{e_z}\cdot\oint\limits_{CS_1}\mathbf{n}\cdot\mathbf{UU}dS=\int\limits_{S_D}u(1-u)dS=\int\limits_{S_D}a(1-a)dS \tag{8}$$

where $u=1-a$ is the axial velocity magnitude, $CS_1$ is the external control surface bounding $CV_1$, and $S_D$ denotes the downwind face of $CV_1$. The $\mathbf{n}\cdot(\frac{1}{2}U^2\mathbf{I})$ term, which we will call the kinetic energy term, contains contributions from all three components of velocity; the thrust contribution from axial velocity is

$$-\frac{1}{2}\int\limits_{S_D}(1-u^2)dS=-\frac{1}{2}\int\limits_{S_D}(2a-a^2)dS. \tag{9}$$

The right-hand sides of equations (8) and (9) can be summed to give

$$-\frac{1}{2}\int\limits_{S_D}a^2dS. \tag{10}$$

The kinetic energy term also has a contribution from the radial velocity, $v$:

$$\frac{1}{2}\int\limits_{S_D}v^2dS, \tag{11}$$

and the contribution from the circumferential velocity is

$$2\int\limits_{S_D}w^2dS, \tag{12}$$

where we have continued the convention to express the circumferential velocity on $S_D$ as $u_\theta=-2w$. The velocities on the lateral cylindrical control surface do not contribute to the kinetic energy term because its normal vector is perpendicular everywhere to $\mathbf{e_z}$.

Vorticity can only be non-zero on the downstream face of the CV and the last two terms on the right-hand side of (5), which we call the vortex terms, simplify to

$$\frac{1}{2}\int\limits_{S_D}(1-a)x\Omega_\theta dS-\int\limits_{S_D}\Omega_z wxdS. \tag{13}$$

Because the radial components of velocity and vorticity are parallel to $\mathbf{x}=x\mathbf{e_r}$ on $S_D$, where $\mathbf{e_r}$ is the radial unit vector, they lend no thrust contribution through the vortex terms.

So, we have seen that the impulse term's thrust contribution vanishes, the two velocity terms in (5) are replaced by the sum of (10) through (12), and the vortex terms lend the contribution given in (13). Summing these contributions, the thrust becomes

$$\frac{T}{\rho}=\frac{1}{2}\Bigg[\int\limits_{S_D}v^2dS-\int\limits_{S_D}a^2dS+4\int\limits_{S_D}w^2dS+\int\limits_{S_D}(1-a)x\Omega_\theta dS-\int\limits_{S_D}2\Omega_z wxdS\Bigg], \tag{14}$$

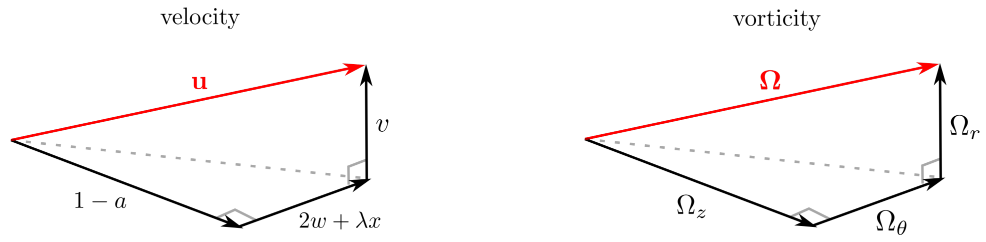


**Figure 2.** Velocity (in the rotating frame of reference) and vorticity vectors in the near wake. If diffusion is neglected and the wake is assumed rigid, velocity and vorticity triangles on each orthogonal plane are geometrically similar.

where all velocities and vorticities are expressed relative to an inertial frame.

Equation (14) has incorporated the assumptions of steady rotation, a rigid wake, and vanishingly small viscosity. To proceed towards the KJ equation for blade-element thrust, let us assume that the effect of diffusion is negligible on $S_D$, such that Kelvin's circulation theorem applies to individual fluid elements. When combined with the assumption of a rigid wake, it is clear that vortex lines must coincide with streamlines in the rotating frame of reference, or else the structure of the vortical wake would appear to deform in that frame. This alignment of streamlines and vortex lines in three dimensions is illustrated in figure 2. On any two-dimensional plane cutting through our three-dimensional volume, the projected velocity and vorticity components generate similar triangles. We are specifically interested in the resulting relationship between axial and circumferential vector components:

$$\frac{2w+\lambda x}{1-a} = \frac{\Omega_\theta}{\Omega_z}. \tag{15}$$

This becomes useful if we rewrite the vortex terms as

$$\frac{1}{2}\int\limits_{S_D} \big[(1-a)\Omega_\theta - (2w+\lambda x)\Omega_z\big] x dS + \frac{1}{2}\int\limits_{S_D} \Omega_z \lambda x^2 dS, \tag{16}$$

because the terms in the square brackets of the first integrand cancel by Equation (15). The remaining term in (16) can be interpreted as a thrust due to wake rotation. We now manipulate this term to express it in terms of velocity. Substituting $dS = x dx d\theta$ into the previous expression, the term becomes

$$\frac{1}{2}\lambda \int\limits_0^{2\pi}\int\limits_0^{\infty} \Omega_z x^3 dx d\theta. \tag{17}$$

The streamwise vorticity in polar coordinates is

$$\Omega_z = -2\left(\frac{dw}{dx} + \frac{w}{x}\right) - \frac{1}{x}\frac{\partial v}{\partial \theta}, \tag{18}$$

where the negative sign on the $w$ terms is due to our chosen sign convection, whereby positive $w$ corresponds to positive torque. When substituted into (17), the contribution of $\partial v/\partial\theta$ vanishes:

$$-\frac{1}{2}\lambda\int_{0}^{\infty}x^{2}\left[\int_{0}^{2\pi}\frac{\partial v}{\partial\theta}d\theta\right]dx=0. \tag{19}$$

The vortex terms thus become

$$-\lambda\int_{0}^{2\pi}\left[\int_{0}^{1}\frac{dw}{dx}x^{3}dx+\int_{0}^{1}wx^{2}dx\right]d\theta. \tag{20}$$

After integrating the first term by parts, and assuming $wx^3\to 0$ as $x\to\infty$, the whole expression becomes

$$2\lambda\int_{0}^{2\pi}\int_{0}^{\infty}wx^{2}dxd\theta=2\lambda\int_{S_D}wxdS. \tag{21}$$

Substituting the expression on the right-hand side into Equation (14), we arrive at

$$\frac{T}{\rho}=\frac{1}{2}\int_{S_D}\left(v^{2}-a^{2}\right)dS+2\int_{S_D}\left(w^{2}+\lambda wx\right)dS. \tag{22}$$

It is now clear that our derivation is approaching Equation (2), the precursor to the KJ thrust equation. The next step is to investigate the relationship between $v$ and $a$, which is done in the next section.

Since a future goal of this research program is unsteady turbine modelling, an alternative derivation of Equation (22) in the rotating frame of reference is offered in Appendix B.

## 4 Implications for Blade Element Momentum Theory

### 4.1 Radial velocity

If the downwind face of the CV is moved to be just ahead of the rotor— replacing integrals on $S_D$ with integrals on $S_U$ in our first use of $CV_2$ shown in figure 1—then $T$ in Equation (22) becomes zero, since no body is enclosed by the CV. The flow on $S_U$ is irrotational, and so the vortex terms in Equation (14) also vanish, and our CV impulse analysis reduces to

$$0=\int_{S_U}\left(v^{2}-a^{2}+u_{\theta}^{2}\right)dS. \tag{23}$$

If there are azimuthal variations of $v$ on $S_U$, then there must be corresponding radial variations in $u_\theta$ to ensure that the axial vorticity remains zero. In the special case of azimuthal uniformity, $u_\theta$ must vanish on $S_U$ so that the circulation around any circle of radius $x$ remains zero (Taylor (1921)). Thus,

$$0=\int_{S_U}\left(v^{2}-a^{2}\right)dS \tag{24}$$

for the circumferentially uniform case. van Kuik (2020) directly corroborates Equation (24), showing it to hold for his model of rotor flow. Since $v$ and $a$ must be continuous across any actuator disk, the equation also holds immediately behind the rotor, but that is the limit to its validity; it cannot hold, for example, in the far wake, where $v$ is everywhere zero but $a \neq 0$.

Equation (24) allows us to infer that $v^2 = a^2$ somewhere on $S_U$ if both $v$ and $a$ are $C_0$-continuous. In a real flow with finite viscosity, we indeed expect both variables to vary continuously, even if we allow for arbitrarily large (but finite) gradients. At the rotor axis, we can assert $v = 0$ by symmetry, but in general $a > 0$ at the axis for a loaded turbine rotor. If $a^2 > v^2$ over the inner rotor, Equation (24) requires that $v^2 > a^2$ over some other part of $S_U$, and the restriction that $v$ and $a$ are continuous then requires $v^2 = a^2$ for at least one radial location. Previous actuator-disk simulations show that $v > a$ in the flow external to the rotor ($x > 1$), and that $v = a$ occurs near the tip of the actuator disk for a wide range of thrust coefficients, e.g. Figures 15 and 16 of Madsen et al. (2010), Figures 3.2 and 3.3 of Sørensen (2016), Figure 4 of Branlard & Meyer Forsting (2020). These results indicate a significant radial deflection of the streamlines in the tip region as they pass through a loaded rotor. We are unaware of any study of the effect of this "crossflow" on blade element forces, but simple expressions for crossflow alterations to airfoil lift and drag are developed by Hodara & Smith (2014).

For a turbine with a finite number of blades, the situation is more complex. The early near-wake measurements of Ebert & Wood (2001) and the recent ones of Eriksen & Krogstad (2017), when converted to co-ordinates rotating with the blades, show large positive and negative values of $v(\theta)$ particularly in the tip region. The circumferential average $v$ or the value at the blades, can, therefore, be small while the average of $v^2$ can be significant. The flow in the tip region of a rotor with a finite number of blades can include motion of the tip vortex towards, rather than away from the axis of rotation, van Kuik et al. (2014).

Assuming that $a$ and $v$ are continuous through the rotor, Equation (24) applies on $S_D$ as well as $S_U$, and Equation (22) simplifies to

$$\frac{T}{\rho} = 2\int\limits_{S_D} \left(w^2 + \lambda w x\right) dS. \tag{25}$$

All that remains to recover Equation (1) is to determine when the blade-element version of (25) is also valid.

### 4.2 Blade-element thrust

To determine the blade element version of (25), we consider two variants of a cylindrical annular CV: one with the downstream face just upwind of the rotor ($V_1$), and one with it just downwind ($V_2$), as shown in figure 3. In the former case, the flow everywhere in the control volume and on the bounding control surface ($S_1$) is irrotational, and since there is no body enclosed by this CV, the thrust is zero and we obtain

$$0 = \oint\limits_{S_1} \mathbf{n} \cdot \left(\frac{1}{2}U^2\mathbf{I} - \mathbf{U}\mathbf{U}\right) dS. \tag{26}$$

As noted above, the circumferential velocity ahead of the rotor vanishes for a circumferentially uniform flow, in which case only radial and axial velocities contribute to the integral in Equation (26). Assuming these velocities to be continuous across

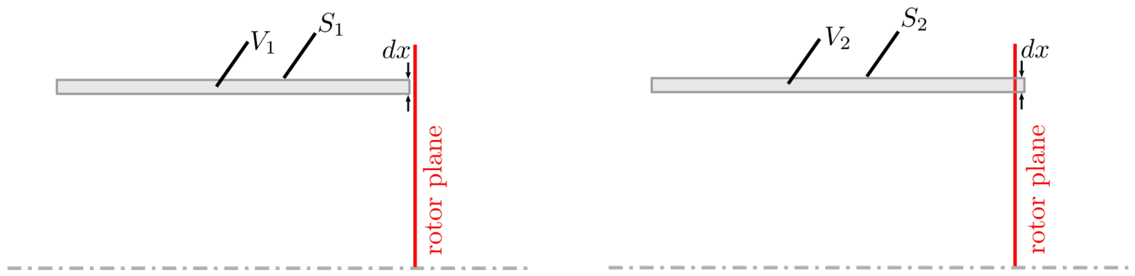


**Figure 3.** Annular CVs used for the blade-element force analysis. The downstream face of width $dx$ is just upstream and just downstream of the rotor for $V_1$ and $V_2$, respectively.

the rotor, their contribution to the same integrand over $S_2$ also vanishes and we obtain

$$\oint_{S_2} \mathbf{n} \cdot \left( \frac{1}{2} U^2 \mathbf{I} - \mathbf{U}\mathbf{U} \right) dS = 2 \int_{S'_D} w^2 dS, \tag{27}$$

where $S'_D$ is the downstream face of the annular CV, $V_2$. Since the trailing vortices pierce $S'_D$, the vortex terms contribute $2\lambda w x$ to the integrand of the thrust integral, as in Equation (22). Vorticity also exists on the lateral boundaries of $V_2$, which we denote $S_L^-$ and $S_L^+$, and the complete thrust contribution of the vortex terms is

$$\frac{1}{2} \mathbf{e_z} \cdot \oint_{S_2} \Big( -\mathbf{n} \cdot \mathbf{U}(\mathbf{x} \times \mathbf{\Omega}) + \mathbf{n} \cdot \mathbf{\Omega}(\mathbf{x} \times \mathbf{U}) \Big) dS = \frac{1}{2} \int_{S_L^-} (v\Omega_\theta - u_\theta \Omega_r) x dS - \frac{1}{2} \int_{S_L^+} (v\Omega_\theta - u_\theta \Omega_r) x dS + 2 \int_{S'_D} \lambda w x dS. \tag{28}$$

We cannot cause the integrals over $S_L^-$ and $S_L^+$ to vanish by arguing that vortex lines and streamlines are coincident (as we did on $S_D$) because diffusion is not neglible in the vicinity of the blades where vorticity is generated. We will nonetheless neglect these contributions without making any positive claim about their insignificance, as this simplification facilitates the recovery the KJ equation. Thereafter combining Equations (27) and (28), the expression for blade-element thrust becomes

$$\frac{1}{\rho} \frac{dT}{dx} = 2 \int_0^{2\pi} \left( w^2 + \lambda w x \right) x d\theta, \tag{29}$$

which is, at last, the same as Equation (1).

Recall that the classical derivation of Equation (29) requires the following assumptions: the trailing wake is thin, i.e. it occupies zero volume; it rotates rigidly with the rotor; and the axial and radial velocities are continuous across the disk. We have arrived at Equation (29) using the latter two of these three assumptions, but we have avoided the thin wake assumption; the new derivation permits the flow to be rotational in the wake. We have, however, invoked the circumferential uniformity assumption early; this was not required in the classical derivation of (29), but is necessary to recover the KJ blade-element thrust equation.

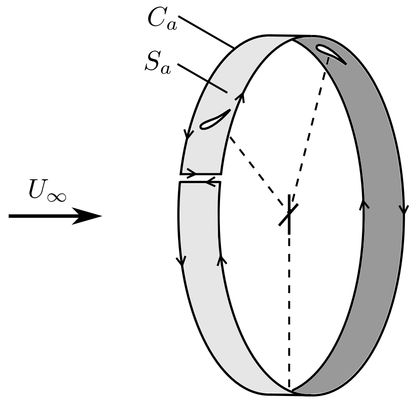


**Figure 4.** The sum of blade circulations at radial station $x$ can be calculated as an area integral of radial vorticity over $S_a$, or a closed line integral over $C_a$, where only the downstream circular section of $C_a$ lends a non-zero contribution.

### 4.3 Kutta-Joukowsky equation

When the flow on $S_D$ is assumed to be circumferentially uniform, Equation (29) becomes

$$\frac{1}{\rho}\frac{dT}{dx} = (w + \lambda x)\int_0^{2\pi} 2wx d\theta. \tag{30}$$

The sum of blade circulations, $\sum_i \Gamma_i$, at a given radial station is equal to the integral on the right-hand side, as can be derived by Stokes' theorem using figure 4. Taking $S_a$ to be a cylindrical surface of radius $x$ intersecting the blades, with $S_a$ bound externally by $C_a$,

$$\sum_i \Gamma_i = \int_{S_a} \Omega_r dS = \oint_{C_a} \mathbf{U} \cdot d\mathbf{l} = \int_0^{2\pi} 2wx d\theta, \tag{31}$$

where $\Omega_r$ is the radial vorticity on $S_a$, $d\mathbf{l}$ is the tangential unit vector along $C_a$, and only the downstream portion of $C_a$ lends a non-zero contribution the contour integral (since azimuthal velocity is zero upstream of the blades, e.g. Taylor (1921)). When all blades are evenly loaded, Equations (30) and (31) combine to yield

$$\frac{1}{\rho}\frac{dT}{dx} = N\Gamma_{BE}(w + \lambda x) \tag{32}$$

where $N$ is the number of blades and $\Gamma_{BE}$ is the bound circulation of the blade element. Equation (32) is the KJ equation for blade-element thrust when $w + \lambda x$ is interpreted as the relative circumferential velocity experienced by the blade element.

Circumferential uniformity is approached as $N \to \infty$, but Equation (32) is approximately true for finite $N$ when $\lambda x \gg w$. This occurs for the power producing region of the blades at sufficiently high $\lambda$.

In blade element analysis, the Kutta-Joukowsky equation is usually introduced as an assumption, but this can only be valid as $N \to \infty$ and/or $\lambda \to \infty$. This result is directly deducible from Equation (29) regardless of the method of its derivation.

Recall that we neglected the contribution of the vortex terms on the lateral boundaries of the annular CV to arrive at Equation (29). Although this assumption led to a result in agreement with the classical approach, this agreement is not sufficient justification. Rather, the classical derivation is simply blind to this effect due to its neglect of vorticity. Future work is needed to investigate the possible significance of the neglected terms.

## 5 Summary and Conclusions

This paper describes the application of vortical impulse theory to determine the thrust on a steadily rotating wind turbine in a steady, spatially uniform wind. The principal attraction of the impulse approach is the absence of pressure in the force equations, which allows them to be applied anywhere in the flow, including immediately behind the rotor. We assume that the vorticity field is steady when viewed by an observer rotating with the blades, so the vortex lines and streamlines in the near wake are aligned in this frame. Subsequently, we rederive the Kutta-Joukowsky (KJ) equation for blade-element thrust under the special condition of circumferential uniformity in the wake (i.e. as the number of blades or the tip speed ratio tends to infinity). Our derivation improves upon the classical derivation — which is based on the Bernoulli equation — by allowing the wake to be rotational and three-dimensional; our derivation is exact for any amount of flow expansion through the rotor. In addition, intermediate steps en route to the KJ equation provide thrust expressions which allow for circumferential non-uniformity and trailing vortex sheets of finite thickness.

The derivation also yields insight into the radial velocity based on Equation (24). When the wake is circumferentially uniform, the magnitudes of the radial velocity and the axial induction factor must be equal somewhere on the plane containing the rotor, and we cite previous studies that show this to occur near the rotor tip across a wide range of thrust coefficients. Equation (24) was verified in the simulations of van Kuik (2020).

An important avenue for future work is an investigation of the effect of vorticity on the lateral boundaries of the annular control volumes. The blade-element KJ equation was recovered only when these terms were neglected, but a general claim of their insignificance has not been made. This complication is not visible to the classical derivation of the KJ equation due to its neglect of vorticity.

## Appendix A: Derivation of blade-element thrust from the unsteady Bernoulli equation

Assume that, at all times, the fluid elements along a streamline passing through the rotor remain irrotational. The trailing vorticity in the wake is assumed to be infinitely thin. The unsteady Bernoulli equation along this streamline is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + u_\theta^2) + p = C \tag{A1}$$

where $C$ is a constant and $\phi$ is the scalar potential. If the wake rotates rigidly with the blades, the unsteady term can be expressed in terms of the circumferential velocity and the rotation rate, $\lambda$:

$$\frac{\partial\phi}{\partial t} = \frac{\partial\theta}{\partial t}\left(\frac{\partial\phi}{\partial\theta}\right) = -\lambda x u_\theta \tag{A2}$$

where the negative sign arises because $\theta$ is evaluated in the moving frame, i.e. $\partial\theta/\partial t = -\lambda$, and the relationship $u_\theta = (1/x)\partial\phi/\partial\theta$ comes directly from the definition of the scalar potential, $\mathbf{U} = \boldsymbol{\nabla}\phi$. We now apply Equation (A1) at locations just up- and downstream of the rotor to solve for the pressure difference across the rotor, $\Delta p$. Assuming radial and axial velocities to be continuous across the disk, only the circumferential velocity contributes to $\Delta p$:

$$\Delta p = \frac{1}{2}u_\theta^2 - \lambda x u_\theta. \tag{A3}$$

Substituting $u_\theta = -2w$, and integrating $\Delta p$ over the rotor, we recover Equation (25):

$$\frac{T}{\rho} = \int\limits_{S_D} \Delta p dS = 2\int\limits_{S_D} \left(w^2 + \lambda w x\right) dS. \tag{A4}$$

Since the Bernoulli equation along any one streamline is independent of neighbouring streamlines, the blade-element version of this equation should also hold, i.e.

$$\frac{1}{\rho}\frac{dT}{dx} = 2\int\limits_0^{2\pi} \left(w^2 + \lambda w x\right) x d\theta, \tag{A5}$$

which is the same as Equations (2) and (29) in the body of the paper.

## Appendix B: The impulse equation for thrust in steady rotating coordinates

When the polar co-ordinates are attached to the blades rotating at $\lambda$, the velocities corresponding to those used in the main text are $(0, \lambda x, 1)$ well upwind, $(v, \lambda x, 1-a)$ immediately upwind of the rotor, and $(v, u'_\theta, 1-a) = (v, 2w + \lambda x, 1-a)$ downwind. Equation (5) requires modification to be expressed in terms of velocities and vorticities evaluated in the rotating frame of reference, $\mathbf{U}'$ and $\boldsymbol{\Omega}'$, respectively. The momentum integral in Equation (4) is replaced by

$$\frac{d}{dt}\int\limits_V \mathbf{U} dV = \frac{d}{dt}\int\limits_V \mathbf{U}' dV + \int\limits_V 2\boldsymbol{\Lambda}\times\mathbf{U}' dV + \int\limits_V \boldsymbol{\Lambda}\times(\boldsymbol{\Lambda}\times\mathbf{x}) dV. \tag{B1}$$

In Noca's derivation of Equation (5) from (4), the pressure, $P$, was removed by using the Navier-Stokes equations in the form of his Equation (3.44):

$$\frac{\partial\mathbf{U}}{\partial t} = -\nabla(P + \frac{1}{2}U^2) + \mathbf{U}\times\boldsymbol{\Omega} + \nabla\cdot\mathbf{T}. \tag{B2}$$

For a steadily rotating CV, (B2) becomes

$$\frac{\partial\mathbf{U}'}{\partial t} = -\nabla(P + \frac{1}{2}U'^2) + \mathbf{U}'\times\boldsymbol{\Omega}' - 2\boldsymbol{\Lambda}\times\mathbf{U}' - \boldsymbol{\Lambda}\times(\boldsymbol{\Lambda}\times\mathbf{x}) \tag{B3}$$

as given, for example, by Equation (3.2.10) of Batchelor (1967). Clearly, the Coriolis term is the penultimate one and the last is the centrifugal. These have now to be included in (5). The right side of Equation (B2) appears explicitly in the second term of Noca's (3.49) as

$$\oint_S \mathbf{x} \times \left( \mathbf{n} \times \left[ \nabla (P + \frac{1}{2} U^2) + \mathbf{U} \times \mathbf{\Omega} + \nabla \cdot \mathbf{T} \right] \right) dS \tag{B4}$$

which means that, in addition to swapping $\mathbf{U}$ for $\mathbf{U}'$ and $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ in (5), use of a noninertial CV requires the extra terms

$$-\oint_S \mathbf{x} \times \left( \mathbf{n} \times \left[ 2\mathbf{\Lambda} \times \mathbf{U}' + \mathbf{\Lambda} \times (\mathbf{\Lambda} \times \mathbf{x}) \right] \right) dS. \tag{B5}$$

The general form of the impulse formulation, Equation (5), in a steadily rotating frame of reference thus becomes

$$\begin{aligned} \frac{\mathbf{F}}{\rho} = -\frac{1}{N-1}\frac{d}{dt}\int_V \mathbf{x} \times \mathbf{\Omega}' dV + \oint_S \mathbf{n} \cdot \left( \frac{1}{2} U'^2 \mathbf{I} - \mathbf{U}'\mathbf{U}' - \frac{1}{N-1}\mathbf{U}'(\mathbf{x} \times \mathbf{\Omega}') + \frac{1}{N-1}\mathbf{\Omega}'(\mathbf{x} \times \mathbf{U}') \right. - \int_V 2\mathbf{\Lambda} \times \mathbf{U}' dV - \\ \int_V \mathbf{\Lambda} \times (\mathbf{\Lambda} \times \mathbf{x}) dV - \frac{1}{N-1}\oint_S \mathbf{x} \times \left( \mathbf{n} \times \left[ 2\mathbf{\Lambda} \times \mathbf{U}' + \mathbf{\Lambda} \times (\mathbf{\Lambda} \times \mathbf{x}) \right] \right) dS + \mathbf{F}_v. \end{aligned} \tag{B6}$$

$\mathbf{F}_v$ will be neglected as before, as we expect viscous effects to be no more important for rotating than stationary wakes. It is assumed that the vortical wake appears stationary relative to the rotating blades, such that the time derivative of impulse — the first integral on the right-hand side of (B6) — vanishes. The Coriolis force can only have radial and circumferential components and the centrifugal force is radial. The radial components cannot contribute to an axial (or indeed to a circumferential) force, and thus the volume integrals of the pseudo-forces at the end of the first line and start of the second line of (B6) do not contribute to $T = \mathbf{e}_z \cdot \mathbf{F}$. The axial component of the centrifugal contribution to the pseudo-force surface integral in the second line of (B6) also vanishes by symmetry, and the remaining Coriolis term can be evaluated using the vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. With $S_D$ denoting the downwind face of the CV, the net contribution, which lies in the axial direction, is

$$-2\int_{S_D} \lambda w x dS \tag{B7}$$

because the term involving $\lambda^2 x^2$ will be equal and opposite at the upwind and downwind faces of the CV.

Thus the reduced form of Equation (B6) is

$$\frac{T}{\rho} = \mathbf{e}_z \cdot \left[ \frac{1}{2}\oint_S \mathbf{n} U'^2 dS - \oint_S \mathbf{n} \cdot \mathbf{U}'\mathbf{U}' dS - \frac{1}{2}\oint_S \mathbf{n} \cdot \mathbf{U}'(\mathbf{x} \times \mathbf{\Omega}') dS + \frac{1}{2}\oint_S \mathbf{n} \cdot \mathbf{\Omega}'(\mathbf{x} \times \mathbf{U}') dS - 2\int_{S_D} \mathbf{e_z} \lambda w x dS \right], \tag{B8}$$

Equation (B8) is now applied to the CVs shown in figure 1, but now rotating at the same $\lambda$ as the rotor. The MD term is the conventional one, derived in any textbook on wind turbine aerodynamics:

$$\int_S u(1-u) dS = \int_{S_D} a(1-a) dS \tag{B9}$$

where $S_D$ denotes the downwind face of the CV. The axial velocity contributes to the second (kinetic energy) term an amount

$$-\frac{1}{2}\int_{S_D}(1-u^2)dS. \tag{B10}$$

Equations (B9) and (B10) can be combined to give

$$-\frac{1}{2}\int_{S_D}a^2dS. \tag{B11}$$

The kinetic energy has a contribution from the radial velocity:

$$\frac{1}{2}\int_{S_D}v^2dS \tag{B12}$$

and the circumferential velocity:

$$\frac{1}{2}\oint_S u_\theta'^2dS = 2\int_{S_D}\left(w^2+\lambda wx\right)dS. \tag{B13}$$

It will be assumed that $w=0$ outside the wake. The radial velocity exiting the horizontal face of the CV at $R_{CV}$ does not contribute to the kinetic energy because $v(R_{CV})\sim 1/R_{CV}$ so the integral of $v^2$ over the cylindrical face can be made arbitrarily small by making $R_{CV}$ sufficiently large.

The vortex terms are better expressed in the inertial frame, $\mathbf{\Omega}=\mathbf{\Omega}'+2\mathbf{\Lambda}$, because non-zero values of $\mathbf{\Omega}$ indicate fluid elements that have experienced viscous shear, whereas the apparent vorticity $\mathbf{\Omega}'$ has no such physical interpretation. The vortex terms are equivalently expressed as

$$\left[-\frac{1}{2}\oint_S\mathbf{n}\cdot\mathbf{U}'(\mathbf{x}\times\mathbf{\Omega})dS+\frac{1}{2}\oint_S\mathbf{n}\cdot\mathbf{\Omega}(\mathbf{x}\times\mathbf{U}')dS\right]-\oint_S\mathbf{n}\cdot\mathbf{\Lambda}(\mathbf{x}\times\mathbf{U}')dS. \tag{B14}$$

For the vortical wake to appear stationary in the rotating frame, $\mathbf{\Omega}$ must be parallel to $\mathbf{U}'$ everywhere that vorticity is non-zero — that is, vortex lines and streamlines are aligned in the rotating frame — by which $\mathbf{n}\cdot\mathbf{\Omega}(\mathbf{x}\times\mathbf{U}')=\mathbf{n}\cdot\mathbf{U}'(\mathbf{x}\times\mathbf{\Omega})$ and the two integrals in square brackets identically cancel, leaving only

$$-\oint_S\mathbf{n}\cdot\mathbf{\Lambda}(\mathbf{x}\times\mathbf{U}')dS=\int_{S_D}2\lambda wxdS \tag{B15}$$

for $T$. Combining these results leads to

$$\frac{T}{\rho}=\frac{1}{2}\int_{S_D}v^2dS-\frac{1}{2}\int_{S_D}a^2dS+2\int_{S_D}\left(w^2+\lambda wx\right)dS+2\int_{S_D}\lambda wxdS-2\int_{S_D}\lambda wxdS \tag{B16}$$

Curiously, three integrals with the integrand $2\lambda wx$ have arisen from three different origins: the kinetic energy term and the vortex terms each yield the same integral in the positive sense, and the Coriolis terms lends an equal and opposite contribution. The choice of which two to cancel is arbitrary, but either choice leads to the final equation:

$$\frac{T}{\rho}=\frac{1}{2}\int_{S_D}\left(v^2-a^2\right)xdx+2\int_{S_D}\left(w^2+\lambda wx\right)dS. \tag{B17}$$

which is identical to Equation (22).

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