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Derivation of an Impulse Equation for Wind Turbine Thrust

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Abstract. Using the concept of impulse in control volume (CV) analysis, we derive a new equation for steady wind turbine thrust in a constant, spatially uniform wind. This equation contains the circumferential velocity and tip speed ratio. We determine the conditions under which the new equation reduces to the standard equation involving only the axial velocity. A major advantage of an impulse formulation is that it removes the pressure and introduces the vorticity, allowing the equation to be used unambiguously immediately behind the blades. Using two CVs with this downwind end - one rotating with the blades, and one stationary - highlights different aspects of the analysis. We assume that the vorticity is frozen relative to an observer rotating with the blades, so the vortex lines follow the local streamlines in the rotating frame and vorticity does not appear explicitly in the impulse equation for force. In the stationary frame, streamlines and vortex lines intersect, and one of the thrust terms can be interpreted as the effect of wake rotation. The impulse analysis also shows the significance of the radial velocity in wind turbine aerodynamics. By assuming both the radial and axial velocities are continuous through the rotor disk, their contributions to the final thrust expression cancel to leave an expression dependent on azimuthal velocity alone. The final integral equation for thrust can be viewed as a generalization of the Kutta-Joukowsky theorem for the rotor forces. We give a proof, for the first time, for the conditions under which the Kutta-Joukowsky equations apply. The analysis is then extended to the blade elements comprising the rotor. The new formulation gives a very simple, exact equation for blade element thrust which is the major contribution of this study. By removing the pressure partly through the kinetic energy contribution of the radial velocity, the new equation circumvents the long-standing concern over the role the pressure forces acting on the expanding annular streamtube intersecting each blade element. It is shown that the necessary condition for blade-element independence of the conventional thrust equation — that which involves the axial induction factor — is the constancy of the vortex pitch in the wake.

1 Introduction

Blade element theory (BET) for wind turbines uses the fundamental assumption that the forces acting on the elements comprising the rotor blades are given by the Kutta-Joukowsky theorem. The thrust and torque are balanced by the change in the axial and angular momentum, respectively, of the flow through a control volume (CV) enclosing the rotor. BET is developed in all texts on wind turbine aerodynamics, such as Burton *et al.* (2011), Wood (2011), and Hansen (2015), so it is unnecessary to repeat it here.

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It has long been recognized that some aspects of BET need improvement. The main example is the failure of the equations in the high thrust region where the thrust coefficient, $C_T = T/(\frac{1}{2}\rho U^2 \pi R^2)$, exceeds unity. T is the rotor thrust, ρ is the density of air, U the wind speed, and R is the blade radius. Part of the reason for the high thrust anomoly may well be the form of the equation for T. It is usually derived by assuming that the gauge pressure in the far-wake is zero, the flow is circumferentially uniform, and the vorticity crossing the CV boundary is not significant. The work described in this paper is part of a longer-term project to address these assumptions using an equation for T derived from an impulse formulation for aerodynamic force. Impulse removes the pressure and introduces vorticity. This allows, for example, the impulse equation to be used in the immediate vicinity of the blades and provides a thrust equation of different form to the conventional one. Because of this, and the fact that most wind energy researchers are unlikely to be familiar with impulse methods, we concentrate in this paper on the derivation of the thrust equation.

Impulse methods have been used to remove the pressure from force balances in fluid mechanics since their introduction by Thomson (1882) to determine the speed of a vortex ring. In their review, Wu et al. (2015) recount that exact impulse-based expressions for aerodynamic force were derived independently by Burgers (1921), Wu (1981) and Lighthill (1986). Discussions of impulse formulations can also be found in Lamb (1932), Batchelor (1967), and Saffman (1992). They have recently gained popularity for use with planar or volume meaurements of fluid velocities (but not pressure) from particle image velocimetry and related techniques, e.g. Rival & van Oudheusden (2017) and Limacher et al. (2018). For example, Limacher et al. (2019a) tested an impulse equation for thrust against measurements of an impulsively started circular cylinder using PIV data, and this method was then compared by Limacher et al. (2019b) to a momentum-based formulation. The "vortical impulse" of a flow is the volume integral of the cross product of the position and vorticity vectors. This study uses the formulation of Noca et al. (1997) which is developed in detail by Noca (1997); we recommend the latter to any reader interested in details. Derivation of the impulse formulation begins with the conventional application of the Reynolds transport theorem to the momentum equation, the application of the so-called "impulse-momentum" identity to replace momentum with vortical impulse (Equation (3.1) of Noca (1997)), and proceeds by removing the pressure using the Euler or Navier-Stokes equations.

Using impulse, we derive the thrust equation for a wind turbine rotating steadily at a tip speed ratio, λ, defined as the ratio of the circumferential velocity of the blade tips to the wind speed. The latter velocity will be used to normalize all velocities. All lengths are normalized by R. Two CVs will be used: the first rotates with the blades, as this unambiguously makes the flow steady, and the second is stationary. We will use cylindrical CVs whose upwind face is well ahead of the blades and whose radius, R_{CV}, is such that $R_{CV} \gg 1$. This ensures that the streamwise velocity along the surface at R_{CV} is equal to the wind speed. Reassuringly, the resulting equations from the two CVs are identical. Part of the reason to include the more complex analysis for a rotating CV is our future intention to analyze unsteady wind turbine aerodynamics, which is probably best formulated in co-ordinates fixed to the rotating blades. Readers who are interested in the results, rather than the details of the derivations, can jump to the identical equations, (19) and (29).

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For this analysis, we assume that the forces acting on the blades and blade elements are generated entirely by vorticity. In other words, we ignore any radial vorticity in the wake resulting from viscous drag. Further, if the circumferential extent of the blade wakes is significant, then the axial velocity will not be continuous across the rotor. The impulse analysis can be extended to this case but that would make a complex introductory description even more complex. In stationary cylindrical polar co-ordinates (x,θ,z), where x is the radius, θ is in the circumferential direction, and z the axial co-ordinate, the velocities sufficiently far upwind of the rotor are $(0,0,1)$. Immediately ahead of the rotor they are $(v,0,1-a)$, where a is the standard axial induction factor, and v is the radial velocity through the rotor. Immediately behind the rotor, the velocities are $(v,2w,1-a)$, where w is the velocity induced at the blades by the wake. When the polar co-ordinates are attached to the blades rotating at λ, the corresponding velocities are$(0,\lambda x,1)$ well upwind, $(v,\lambda x,1-a)$ immediately upwind of the rotor, and $(v,u'_\theta,1-a) = (v,2w+\lambda x,1-a)$ downwind. The components of vorticity are $(0,\Omega_\theta,\Omega_z) = (0,\Omega x/\sqrt{p^2+x^2},\Omega p/\sqrt{p^2+x^2})$ in absolute co-ordinates, where p is the vortex pitch. In rotating co-ordinates, the components are $(0,\Omega x/\sqrt{p^2+x^2},\Omega p/\sqrt{p^2+x^2}-2\lambda)$. In contrast, the velocities of the trailing helical vortices that comprise the wake are $(v,2w+\lambda x,1-a)$ in a stationary CV and $(v,2w,1-a)$ in a rotating one.

Impulse equations are derived from conventional CV analysis based on the Reynolds transport theorem. For an inertial CV in an incompressible fluid, the equation for force, $\mathbf{F}$, is given in all fluid mechanics texts and by Noca's (1997) Equation (3.36):

$$\frac{\mathbf{F}}{\rho} = -\frac{d}{dt}\int_V \mathbf{U}dV + \oint_S \mathbf{n}\cdot\left(-\frac{P}{\rho}\mathbf{I} - \mathbf{UU} + \mathbf{T}\right)dS \tag{1}$$

where t is time, and $\mathbf{U}$ is the velocity vector. V indicates the CV, and S the CV surface. $\mathbf{n}$ is the outward facing normal on S, $\mathbf{I}$ is the unit tensor, P is the pressure, and the next term gives the conventional momentum deficit (MD) when the equation is used to determine thrust. $\mathbf{T}$ is the viscous stress tensor. By removing P using the Navier-Stokes equations, Noca obtained his Equations (3.55) and (3.56), which we combine as

$$\frac{\mathbf{F}}{\rho} = -\frac{1}{N-1}\frac{d}{dt}\int_V \mathbf{x}\times\mathbf{\Omega}dV + \oint_S \mathbf{n}\cdot\left(\frac{1}{2}U^2\mathbf{I} - \mathbf{UU} - \frac{1}{N-1}\mathbf{Ux}\times\mathbf{\Omega} + \frac{1}{N-1}\mathbf{\Omega}(\mathbf{x}\times\mathbf{U}) + \frac{1}{N-1}[\mathbf{x}\cdot(\nabla\cdot\mathbf{T})\mathbf{I} - \mathbf{x}(\nabla\cdot\mathbf{T})] + \mathbf{T}\right)dS \tag{2}$$

where $N=3$ (for our analysis) is the dimension, $\mathbf{\Omega}$ is the vorticity vector, and $\frac{1}{2}U^2 = \frac{1}{2}\mathbf{U}\cdot\mathbf{U}$ is the kinetic energy per unit mass. In common with other CV analyses of wind turbines, all terms in $\mathbf{T}$ will be ignored. Note that the conventional MD term appears in both (1) and (2). Of major importance for wind turbine aerodynamics is that the removal of P introduces the vorticity. For unsteady flow, the first term on the right is the time derivative of the impulse. This term will vanish for the steady flows analyzed here. We will, however, continue to describe our analysis as an "impulse" one because it follows the same general path, and because, though not immediately obvious, the remaining terms are related to the impulse in the domain outside of V (Wu et al. (2015), Kang et al. (2017)). The two vortex efflux terms may be non-zero in both steady and unsteady flow.

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The steady version of (2) is applied in the next section to a CV rotating with the blades, and, in the subsequent section, to a stationary CV. Angular momentum and angular impulse are discussion in section 4, and then the consequences of the present impulse analysis for BET are discussed in section 5. Finally, our conclusions are presented in section 6.

2 The Impulse Equation for Thrust in Steady Rotating Co-ordinates

The cylindrical CV of radius R_{CV} begins sufficiently far from the rotor that no influence of the rotor occurs on the inlet face. The downwind exit face is somewhere behind the rotor. Because the CV is rotating, Equation (2) requires modification to be expressed in terms of velocities and vorticities evaluated in the frame of reference fixed to the rotor, $\mathbf{U}'$ and $\mathbf{\Omega}'$, respectively. The momentum integral in equation (1) is replaced by

$$\frac{d}{dt}\int\limits_V \mathbf{U}dV = \frac{d}{dt}\int\limits_V \mathbf{U}'dV + \int\limits_V 2\mathbf{\Lambda}\times\mathbf{U}'dV + \int\limits_V \mathbf{\Lambda}\times(\mathbf{\Lambda}\times\mathbf{x})dV. \tag{3}$$

In Noca's derivation of Equation (2) from (1), the pressure, P, was removed by using the Navier-Stokes equations in the form of his Equation (3.44):

$$\frac{\partial\mathbf{U}}{\partial t} = -\nabla(P+\frac{1}{2}U^2)+\mathbf{U}\times\mathbf{\Omega}+\nabla\cdot\mathbf{T}. \tag{4}$$

For a steadily rotating CV, (4) becomes

$$\frac{\partial\mathbf{U}'}{\partial t} = -\nabla(P+\frac{1}{2}U'^2)+\mathbf{U}'\times\mathbf{\Omega}'-2\mathbf{\Lambda}\times\mathbf{U}'-\mathbf{\Lambda}\times(\mathbf{\Lambda}\times\mathbf{x}) \tag{5}$$

as given, for example, by Equation (3.2.10) of Batchelor (1967). $\mathbf{\Lambda}$ is the rotation vector; for steady rotation; $\mathbf{\Lambda}=\lambda\mathbf{e_z}$ where $\mathbf{e_z}$ is the unit vector in the $z-$direction. Clearly, the Coriolis term is the penultimate one and the last is the centrifugal. These have now to be included in (2). The right side of Equation (4) appears explicitly in the second term of Noca's (3.49) as

$$\oint\limits_S \mathbf{x}\times\left(\mathbf{n}\times\left[\nabla(P+\frac{1}{2}U^2)+\mathbf{U}\times\mathbf{\Omega}+\nabla\cdot\mathbf{T}\right]\right)dS \tag{6}$$

which means that, in addition to swapping $\mathbf{U}$ for $\mathbf{U}'$ and $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ in (2), use of a noninertial CV requires the extra terms

$$-\oint\limits_S \mathbf{x}\times\left(\mathbf{n}\times\left[2\mathbf{\Lambda}\times\mathbf{U}'+\mathbf{\Lambda}\times(\mathbf{\Lambda}\times\mathbf{x})\right]\right)dS. \tag{7}$$

The general form of the impulse formulation in a steadily rotating frame of reference thus becomes

$$
\begin{aligned}
\frac{\mathbf{F}}{\rho} = -\frac{1}{N-1}\frac{d}{dt}\int\limits_V \mathbf{x}\times\mathbf{\Omega}' dV + \oint\limits_S \mathbf{n}\cdot\Bigg(&\frac{1}{2}U'^2\mathbf{I} - \mathbf{U}'\mathbf{U}' - \frac{1}{N-1}\mathbf{U}'\mathbf{x}\times\mathbf{\Omega}' + \frac{1}{N-1}\mathbf{\Omega}'(\mathbf{x}\times\mathbf{U}') + \\
&\frac{1}{N-1}[\mathbf{x}\cdot(\nabla\cdot\mathbf{T}')\mathbf{I} - \mathbf{x}(\nabla\cdot\mathbf{T}')] + \mathbf{T}'\Bigg)dS. \\
&-\int\limits_V 2\mathbf{\Lambda}\times\mathbf{U}'dV - \int\limits_V \mathbf{\Lambda}\times(\mathbf{\Lambda}\times\mathbf{x})dV \\
&-\frac{1}{N-1}\oint\limits_S \mathbf{x}\times\Big(\mathbf{n}\times\Big[2\mathbf{\Lambda}\times\mathbf{U}' + \mathbf{\Lambda}\times(\mathbf{\Lambda}\times\mathbf{x})\Big]\Big)dS.
\end{aligned} \tag{8}
$$

Henceforth, viscous stresses on S will be neglected, eliminating the second line of Equation (8), as we expect them to be no more important for rotating than stationary wakes. It is assumed that the vortical wake appears stationary relative to the rotating blades, such that the time derivative of impulse — the first integral on the right-hand side of (8) — vanishes. The Coriolis force can only have radial and circumferential components and the centrifugal force is radial. The radial components cannot contribute to an axial (or indeed to a circumferential) force, and thus the volume integrals of the pseudo-forces on the third line of (8) do not contribute to $T = \mathbf{e}_z\cdot\mathbf{F}$. The axial component of the centrifugal contribution to the pseudo-force surface integral on the fourth line of (8) also vanishes by symmetry, and the remaining Coriolis term can be evaluated using the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. With S_D denoting the downwind face of the CV, the net contribution, which lies in the axial direction, is

$$
-2\int\limits_{S_D} \lambda w x dS \tag{9}
$$

because the term involving $\lambda^2 x^2$ will be equal and opposite at the upwind and downwind faces of the CV.

Thus the reduced form of Equation (8) is

$$
\frac{T}{\rho} = \mathbf{e}_z\cdot\left[\frac{1}{2}\oint\limits_S \mathbf{n}U'^2 dS - \oint\limits_S \mathbf{n}\cdot\mathbf{U}'\mathbf{U}'dS - \frac{1}{2}\oint\limits_S \mathbf{n}\cdot\mathbf{U}'(\mathbf{x}\times\mathbf{\Omega}')dS + \frac{1}{2}\oint\limits_S \mathbf{n}\cdot\mathbf{\Omega}'(\mathbf{x}\times\mathbf{U}')dS - 2\int\limits_{S_D}\mathbf{e_z}\lambda w x dS\right], \tag{10}
$$

Equation (10) is now applied to a CV rotating at the same λ as the rotor. The MD term is the conventional one, derived in any textbook on wind turbine aerodynamics:

$$
\int\limits_S u(1-u)dS = \int\limits_{S_D} a(1-a)dS \tag{11}
$$

where S_D denotes the downwind face of the CV. The axial velocity contributes to the second (kinetic energy) term an amount

$$
-\frac{1}{2}\int\limits_{S_D}(1-u^2)dS. \tag{12}
$$

Equations (11) and (12) can be combined to give

$$
-\frac{1}{2}\int\limits_{S_D} a^2 dS. \tag{13}
$$

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The kinetic energy has a contribution from the radial velocity:

$$\frac{1}{2}\int\limits_{S_D} v^2 dS \tag{14}$$

and the circumferential velocity:

$$\frac{1}{2}\oint\limits_{S} u_\theta'^2 dS = 2\int\limits_{S_D}(w^2+\lambda wx)dS. \tag{15}$$

It will be assumed that $w=0$ outside the wake. The radial velocity exiting the horizontal face of the CV at R_{CV} does not contribute to the kinetic energy because $v(R_{CV})\sim 1/R_{CV}$ so the integral of v^2 over the cylindrical face can be made arbitrarily small by making R_{CV} sufficiently large.

The vortex terms are better expressed in the inertial frame, $\mathbf{\Omega}=\mathbf{\Omega}'+2\mathbf{\Lambda}$, because non-zero values of $\mathbf{\Omega}$ indicate fluid elements that have experienced viscous shear, whereas the apparent vorticity $\mathbf{\Omega}'$ has no such physical interpretation. The vortex terms are equivalently expressed as

$$\left[-\frac{1}{2}\oint\limits_{S}\mathbf{n}\cdot\mathbf{U}'(\mathbf{x}\times\mathbf{\Omega})dS+\frac{1}{2}\oint\limits_{S}\mathbf{n}\cdot\mathbf{\Omega}(\mathbf{x}\times\mathbf{U}')dS\right]-\oint\limits_{S}\mathbf{n}\cdot\mathbf{\Lambda}(\mathbf{x}\times\mathbf{U}')dS. \tag{16}$$

For the vortical wake to appear stationary in the rotating frame, $\mathbf{\Omega}$ must be parallel to $\mathbf{U}'$ everywhere that vorticity is non-zero — that is, vortex lines and streamlines are aligned in the rotating frame — by which $\mathbf{n}\cdot\mathbf{\Omega}(\mathbf{x}\times\mathbf{U}')=\mathbf{n}\cdot\mathbf{U}'(\mathbf{x}\times\mathbf{\Omega})$ and the two integrals in square brackets identically cancel, leaving only

$$-\oint\limits_{S}\mathbf{n}\cdot\mathbf{\Lambda}(\mathbf{x}\times\mathbf{U}')dS=\int\limits_{S_D}2\lambda wx dS \tag{17}$$

for T. Combining these results leads to

$$\frac{T}{\rho}=\frac{1}{2}\int\limits_{S_D}v^2dS-\frac{1}{2}\int\limits_{S_D}a^2dS+2\int\limits_{S_D}(w^2+\lambda wx)dS+2\int\limits_{S_D}\lambda wx dS-2\int\limits_{S_D}\lambda wx dS \tag{18}$$

Curiously, three integrals with the integrand $2\lambda wx$ have arisen from three different origins: the kinetic energy term and the vortex terms each yield the same integral in the positive sense, and the Coriolis terms lends an equal and opposite contribution. The choice of which two to cancel is arbitrary, but either choice leads to the final equation:

$$\frac{T}{\rho}=\frac{1}{2}\int\limits_{S_D}v^2dS-\frac{1}{2}\int\limits_{S_D}a^2dS+2\int\limits_{S_D}(w^2+\lambda wx)dS. \tag{19}$$

3 The Impulse Equation for Thrust in Stationary Co-ordinates

Within an absolute frame of reference, one can likewise arrive at Equation (19). The time derivative of the impulse integral in (2) does not necessarily vanish identically as it did in the rotating frame, but its thrust contribution does; since the vortical

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wake rotates with the blades, the magnitude of the impulse integral cannot change, and its time derivative reduces to

$$\frac{d}{dt}\int\limits_V \mathbf{x}\times\mathbf{\Omega}dV = \mathbf{\Lambda}\times\int\limits_V \mathbf{x}\times\mathbf{\Omega}dV, \tag{20}$$

whose thrust contribution vanishes due to the parallelism of $\mathbf{\Lambda}$ and $\mathbf{e_z}$. The axial components of the remaining terms in Equation (2) reduce to

$$\frac{T}{\rho} = \pi\int\limits_{S_D} v^2 dS - \pi\int\limits_{S_D} a^2 x dx + 4\pi\int\limits_{S_D} w^2 dS + \frac{1}{2}\int\limits_{S_D}(1-a)x\Omega_\theta dS - \frac{1}{2}\int\limits_{S_D} 2\Omega_z w x dS, \tag{21}$$

where all velocities and vorticities are expressed relative to an inertial frame. As already stated, the vortex lines coincide with streamlines in the rotating frame of reference, yielding the relationships

$$\frac{2w+\lambda x}{1-a} = \frac{\Omega_\theta}{\Omega_z} = \frac{x}{p}, \tag{22}$$

where p is the vortex pitch. Thus, the vortex terms can be rewritten as

$$\frac{1}{2}\int\limits_{S_D}\left[(1-a)\Omega_\theta - (2w+\lambda x)\Omega_z\right] x dS + \frac{1}{2}\int\limits_{S_D}\Omega_z \lambda x^2 dS. \tag{23}$$

The terms in the square brackets of the first integrand cancel by Equation (22). The vortex terms thus reduce to the remaining term in (23), which can be interpreted as a thrust due to wake rotation. We now proceed to manipulate this term so as to recover Equation (19). Substituting $dS = x dx d\theta$ into the previous expression, the term becomes

$$\frac{1}{2}\lambda\int\limits_0^{2\pi}\int\limits_0^{\infty}\Omega_z x^3 dx d\theta. \tag{24}$$

The streamwise vorticity in polar coordinates is

$$\Omega_z = -2\left(\frac{dw}{dx} + \frac{w}{x}\right) - \frac{1}{x}\frac{\partial v}{\partial\theta}. \tag{25}$$

When substituted into (24), the contribution of $\partial v/\partial\theta$ vanishes:

$$-\frac{1}{2}\int\limits_0^1 x^2\left[\int\limits_0^{2\pi}\frac{\partial v}{\partial\theta}d\theta\right]dx = 0. \tag{26}$$

The vortex terms thus become

$$-\lambda\int\limits_0^{2\pi}\left[\int\limits_0^1 \frac{dw}{dx}x^3 dx + \int\limits_0^1 w x^2 dx\right]d\theta. \tag{27}$$

After integrating the first term by parts, and assuming $wx^3 \to 0$ as $x \to \infty$, the whole expression becomes

$$2\lambda\int\limits_0^{2\pi}\int\limits_0^{\infty} w x^2 dx d\theta = 2\lambda\int\limits_{S_D} w x dS. \tag{28}$$

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Replacing the vortex terms in equation (21) with this expression, we arrive at

$$\frac{T}{\rho} = \frac{1}{2}\int\limits_{S_D} v^2 x dx - \frac{1}{2}\int\limits_{S_D} a^2 dS + 2\int\limits_{S_D} \left(w^2 + \lambda w x\right) dS, \tag{29}$$

which is identical to Equation (19).

4 Angular Momentum and Angular Impulse

The other dynamic conservation equation applied in conventional CV analysis of wind turbines is for angular momentum. In general vector form, this is expressed as

$$\frac{\boldsymbol{\tau}}{\rho} = -\frac{d}{dt}\int\limits_{V_m} \mathbf{x}\times\mathbf{U} dV \tag{30}$$

$$= -\frac{d}{dt}\int\limits_{V} \mathbf{x}\times\mathbf{U} dV - \oint\limits_{S} \mathbf{n}\cdot\mathbf{U}(\mathbf{x}\times\mathbf{U}) dS \tag{31}$$

where V_m is a material volume, and $\boldsymbol{\tau} = \tau\mathbf{e_z}$ is the torque on the rotor. Assuming the velocity field in the CV to be stationary relative to the blades, the axial component of the first integral vanishes, as previously shown for the impulse derivative term using Equation (20). Equation (31) for the rotor torque then gives the conventional result:

$$\frac{\tau}{\rho} = 2\int\limits_{S_D} (1-a) x w dS. \tag{32}$$

Angular momentum conservation can also be expressed in terms of angular impulse, but, as we will briefly demonstrate, this formulation is unnecessarily complicated. Since the standard angular momentum CV analysis permits the CV exit to be placed directly behind the rotor (due to the absence of a pressure contribution to torque), invoking impulse is of no benefit.

The identity relating angular momentum to angular impulse, given in Equation (3.3.9) of Wu et al. (2015), can be differentiated with respect to time to yield

$$\frac{d}{dt}\int\limits_{V_m} \mathbf{x}\times\mathbf{U} dV = -\frac{1}{2}\frac{d}{dt}\int\limits_{V_m} |\mathbf{x}|^2 \boldsymbol{\Omega} dV + \frac{1}{2}\frac{d}{dt}\oint\limits_{S_m} |\mathbf{x}|^2 \mathbf{n}\times\mathbf{U} dS, \tag{33}$$

where S_m is the boundary of the material volume, and the first term on the right-hand side is the time derivative of angular impulse. It is tempting to assume that the last integral will vanish, and that the integral on the right-hand side of Equation (30) can be replaced by the derivative of angular impulse, expressing the torque in terms of the trailing vorticity crossing S_D. Substitution of the Navier-Stokes equation for the material derivative of $\mathbf{U}$ in the last integral of (33) will yield an integrand of the form $|\mathbf{x}|^2\mathbf{n}\times\nabla P$, whose integral will vanish on the closed contour of an axisymmetric CV, and viscous effects can be neglected as before. Deformation of the material surface, however, gives rise to derivatives of $|\mathbf{x}|^2$ and $\mathbf{n}$ that do not vanish in general. In the special case that axial velocity is constant on the downstream face S_D, the second integral does vanish and Equation (32) can be recovered from the angular impulse term after integration by parts. However, the assumption of constant

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axial velocity on S_D was unnecessary in the standard angular momentum approach, and we thus find that the concept of angular impulse has not enriched the analysis.

5 Impulse and Blade Element Theory

If the downwind face of the CV is moved to be just ahead of the rotor, T in Equation (19) becomes zero as does the last term involving w, Taylor (1921). Thus

$$0 = \int\limits_{S_D} \left(v^2 - a^2\right) dS. \tag{34}$$

For any actuator disk, Equation (34) implies that the magnitude of v and a must be equal over at least some of the wake. More specifically, it can be shown that the integral over the wake is negative and positive over the external flow, where $v > a$. As a likely consequence, $v \approx a$ at the edge of the wake, as shown by Figure 15 of Madsen et al. (2010) and Figure 3.2 of Sørensen (2016). This implies a significant radial deflection of the streamlines in the tip region as they pass through the rotor, but we are unaware of any study of the effect of this "crossflow" on blade element forces. Simple expressions for crossflow alterations to airfoil lift and drag are developed by Hodara & Smith (2014). For a turbine with a finite number of blades, the situation is more complex. The early near-wake measurements of Ebert & Wood (2001) and the recent ones of Eriksen & Krogstad (2017) show large positive and negative values of $v(\theta)$ particularly in the tip region. The circumferential average v or the value at the blades, can, therefore, be small while the average of v^2 can be significant.

Assuming that a and v are continuous through the rotor, Equation (34) applies when the CV face is moved immediately behind the rotor so that

$$\frac{T}{\rho} = 2 \int\limits_{S_D} \left(w^2 + \lambda w x\right) dS. \tag{35}$$

To determine the blade element version of (35), we consider two variants of an annular CV with lateral surfaces coinciding with mean streamsurfaces: one with the downstream face just upwind of the rotor (V_1), and one with it just downwind (V_2). In the former case, the flow everywhere in the control volume and on the bounding control surface (S_1) is irrotational, and since there is no body enclosed by this CV, we obtain

$$\oint\limits_{S_1} \mathbf{n} \cdot \left(\frac{1}{2} U^2 \mathbf{I} - \mathbf{U}\mathbf{U}\right) dS = 0. \tag{36}$$

Assuming the radial and axial velocities are continuous across the rotor, their contribution to the same integrand over S_2 also vanishes, yielding a thrust contribution dependent on azimuthal velocity alone:

$$\oint\limits_{S_2} \mathbf{n} \cdot \left(\frac{1}{2} U^2 \mathbf{I} - \mathbf{U}\mathbf{U}\right) dS = 2 \int\limits_{S_{D2}} w^2 dS, \tag{37}$$

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where S_{D2} is the downstream face of V_2. Since the trailing vortices pierce S_{D2}, the vortex terms contribute $2\lambda wx$ to the integrand of the thrust integral, as in equations (19) and (29), to give the expression for the thrust on a blade element as

$$\frac{1}{\rho}\frac{dT}{dx} = 2\int_{0}^{2\pi} \left(w^2 + \lambda wx\right) x d\theta. \tag{38}$$

An equivalent result to Equation (38) was obtained by Sørensen (2016) in the context of generalized momentum theory. In the derivation of his Equation (4.6), velocities are evaluated in the rotating frame of reference, and the Bernoulli constant, as evaluated in this frame, is assumed continuous across the rotor. Although his approach is difficult to justify, the more rigorous derivation presented here affirms his result. It is also worth emphasizing that Equation (38) is valid for azimuthally non-uniform flows.

Equation (38) is the most important result in this paper because it shows that there is no contribution from the pressure to dT/dx. It was argued by Goorjian (1972), and many subsequent authors, e.g, Sørensen (2016), that the standard blade element momentum balance is "invalid" because it lacks a pressure term arising from the expansion of the streamtubes. The exact impulse form of the blade element thrust equation has nothing equivalent to a pressure term, and thus does not suffer the limitation of the conventionally derived thrust equation. The present result does not imply the general independence of blade elements in the manner usually assumed in BET; any variation in pitch (p) across the wake means that the axial velocity at blade element position x_{BE} is determined partly by the vortex structure for $x > x_{BE}$, and not solely by the local thrust contribution. The reason is that the average u within a helical vortex is proportional to $1/p$, Kawada, (1936), Hardin (1982), and thus blade element independence requires p to be constant across the wake. In the Appendix, we provide an alternative derivation of Equation (38) which gives further information on the role of the pressure on the expanding annular streamtubes.

If the velocities behind the blades are due entirely to the bound vorticity of the blades and the trailing vorticity, with viscous traction on the blade having little effect on the trailing vorticity field, then (35) can, in some circumstances, be written as

$$\frac{1}{\rho}\frac{dT}{dx} = N\Gamma_{BE}\left(w + \lambda x\right) \tag{39}$$

where N is the number of blades and Γ_{BE} is the bound circulation of the blade element. Equation (39) requires at least two conditions to be valid. The first is that the blade elements behave equally, i.e. Γ_{BE} is the same for all elements. Second, the circumferential average of w^2 must equal the square of the circumferential average of w which becomes true as $N \to \infty$. Alternatively, if $\lambda x \gg w$ any circumferential non-uniformity in w becomes relatively small. This occurs for the power producing region of the blades at sufficiently high λ. Equation (32) can also be rewritten in terms of Γ_{BE} with similar restrictions on the validity of the resulting equation:

$$\frac{1}{\rho}\frac{d\tau}{dx} = N\Gamma_{BE}(1-a)x. \tag{40}$$

Equations (39) and (40) are the usual form of the Kutta-Joukowsky theorem for the forces on a blade element. We are unaware of any similar proof of their validity or of the restrictions on that validity; in blade element analysis, the Kutta-Joukowsky equation is usually introduced as an assumption.

Under certain conditions, Equation (38) reduces to the classical expression for blade-element thrust expressed in terms of a rather than w. Using Equation (22), and assuming azimuthal uniformity, Equation (38) becomes

$$\frac{1}{\rho}\frac{dT}{dx} = 4\pi x\left[(1-a)\frac{wx}{p} - w^2\right]. \tag{41}$$

If p is constant across the wake, it follows from the analysis of a single helical vortex by Kawada (1936) and Hardin (1982) that

$$\frac{x}{p} = \frac{a}{w}, \tag{42}$$

which, after substitution into Equation (41), gives the conventional thrust coefficient, C_T, as

$$\frac{dC_T}{dx} = 8[a(1-a) - w^2]x \rightarrow 8a(1-a)x \quad \text{as} \quad \lambda \rightarrow \infty \tag{43}$$

where the asymptotic form is commonly used in BET for any λ. The usual derivation of (43) assumes that there is no contribution from the pressure to the momentum balance in annular CVs, which is equivalent to assuming that the forces on blade elements can be evaluated independently. The present derivation of Equation (43) shows that a necessary condition for blade-element independence is constant p across the wake, which can occur even in the presence of radial pressure gradients. Fortunately, a constant-p wake defines an optimal rotor, since an optimal rotor is expected to produce a rigid helicoidal wake translating at constant speed (originally from Betz (1919), as cited in Okulov and Sørensen (2008)). However, this does not necessarily hold for sub-optimal operating conditions, for which performance evaluation remains practically necessary. One solution to this problem would be to modify u in Equation (43) to account for variations in $p(x)$ for $x > x_{BE}$. This could be especially relevant for diffuser-augmented turbines, where optimality and constant pitch may not coincide.

6 Summary and Conclusions

This paper describes the application of the impulse formulation of Noca (1997) to determine the torque and thrust on a steadily rotating wind turbine in a steady, spatially uniform wind. The impulse analysis starts with the usual Reynolds transport theorem applied to the momentum equation for a finite CV and proceeds by removing the pressure. This has two major attractions for wind turbine analysis: (a) the resulting equation can be applied easily anywhere in the flow and (b) the role of the trailing vorticity becomes clear. We assume that the vorticity field is steady when viewed by an observer rotating with the blades, so the vortex lines and streamlines are aligned in this frame. This alignment simplifies the contribution of the trailing vorticity to the thrust expression, and the resulting term can be interpreted as a contribution due to the rotation of the vortical wake. By assuming the axial velocity to be continuous across the rotor, we also assume that the blade forces are generated entirely by vorticity, so that, for example, the viscous drag of the elements is ignored. We consider the thrust equation both for the whole rotor and for the individual blade elements. The equation for the latter shows that the radial velocity must have the same magnitude as the axial induction factor over at least some of the wake and exceeds the factor in the external flow.

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Probably the most significant result of the present analysis is for the differential form of the thrust equation for the blade elements comprising the rotor:

$$\frac{1}{\rho}\frac{dT}{dx} = 2\int_0^{2\pi} \left(w^2 + \lambda w x\right) x d\theta$$

which is Equation (38) reproduced here because of its importance. The absence of a pressure contribution in this equation is exact, rather than approximate as in the case of the conventional thrust equation used in blade-element theory. The elimination of pressure effects establishes precisely the condition for blade element independence of the conventional thrust equation, which is the constancy of the vortex pitch across the wake. Note, however, that the thrust expression derived herein is sufficiently general to account for wakes of non-constant pitch, and it remains valid for finite-bladed rotors exhibiting azimuthal non-uniformity in the wake. Under the auxiliary conditions that the number of blades and/or the tip speed ratio is sufficiently large, we proved that the Kutta-Joukowsky equation gives the blade element forces, as is assumed in standard blade element analysis.

Appendix A: An alternative derivation of the blade element thrust equation

We present here an alternative derivation of the blade element thrust equation that highlights another aspect of the role of the pressure on the expanding streamsurfaces. The contributions to the impulse equation for elemental thrust, dT, are area integrals over the upwind and downwind faces of the CV V_2 used in Section 5. The former can easily be rewritten in terms of the latter using continuity. In addition to the blade element versions of the two $w-$containing terms in (19), there are contributions to the integrand from:

- The kinetic energy due to the wind speed entering the inlet face of the CV with the value $-\frac{1}{2}(1-a)$,
- The kinetic energy leaving the CV due to a and v:

  $$\frac{1}{2}\left((1-a)^2 + v^2\right),$$

- The usual MD term for the annular streamtube, given by

  $$(1-a) - (1-a)^2 = a(1-a),$$

  and
- The component of kinetic energy normal to the underside and upperside of the annular streamtube.

Only the last is not straightforward. Consider a new CV bounded by the streamsurface passing through the rotor at radius x_{BE} and extending to R_{CV} as before. This CV is outside V_1 used in Section 5. There is no thrust and the impulse thrust equation

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reduces to

$$\int_0^{2\pi}\int_{x_{BE}}^{\infty}\left(v^2-a^2\right)x\,dx\,d\theta=-\int_0^{2\pi}\int_0^{x_{BE}}\left(v^2-a^2\right)x\,dx\,d\theta=\int_0^{2\pi}\int_{x_{0,BE}}^{x_{BE}}\left(1-(1-a)^2-v^2\right)x\frac{dx}{dz}dz\,d\theta \tag{A1}$$

where $x_{0,BE}$ is the radius in the undisturbed upwind flow of the streamsurface passing through x_{BE}. The last integrand is evaluated along the streamsurface and must be positive, as must its integral which also gives the non-zero axial component of the pressure acting on the streamsurface. Therefore, the area integral of v^2-a^2 is *not* zero over the wake; S_D must extend to $R_{CV} \gg 1$ for (34) to hold. Applying (A1) to the upper and lower surfaces of the streamtube, and combining the result with the other three contributions to dT, all terms in a and v cancel to recover our main result, Equation (38).

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References

Batchelor, G.K.: An Introduction to Fluid Dynamics, C.U.P., 1967.

Betz A. Schraubenpropeller mit Geringstem Energieverlust. Dissertation, Gottingen Nachrichten, Gottingen, 1919.

Burgers, J. On the resistance of fluids and vortex motion. Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences, 23, 774–782, 1921.

Burton, T., Jenkins, N., Sharpe, D.,& Bossanyi, E. Wind Energy Handbook. John Wiley & Sons, 2011.

Ebert, P. R., Wood, D. H.: The near wake of a model horizontal-axis wind turbine: Part 3: properties of the tip and hub vortices. Renewable Energy, 22(4), 461-472, 2001.

Eriksen, P. E., Krogstad, P. Å.: An experimental study of the wake of a model wind turbine using phase-averaging. International Journal of Heat and Fluid Flow, 67, 52-62, 2017.

Goorjian, P.M.: An invalid equation in the general momentum theory of the actuator disc. AIAA Journal, 10(4): 543-544, 1972.

Hansen, M. O. Aerodynamics of Wind Turbines. Routledge, 2015.

Hardin, J.C.: The velocity field induced by a helical vortex filament. Phys. Fluids, 25, 1949-1952, 1982.

Hodara, J., Smith, M. J.: Improvement of crossflow aerodynamic predictions for forwarded flight at all advance ratios, 40th European Rotorcraft Forum, Southampton, UK, September 2–5, 2014.

Kang, L., Liu, L., Su, W., Wu, Jiezhi. A minimum-domain impulse theory for unsteady aerodynamic force with discrete wake. Theoretical and Applied Mechanics Letters 7, 306-310, 2017.

Kawada, S.: Induced velocity by helical vortices. J. Aeronaut. Sci. 3, 86-87, 1936.

Lamb, H.: Hydrodynamics, Dover, 6th edition, 1932.

Lighthill, J.: An informal introduction to theoretical fluid mechanics, Oxford University Press, New York, 1986.

Limacher, E., Morton, C., Wood, D.: Generalized derivation of the added-mass and circulatory forces for viscous flows, Physical Review Fluids, 3, 014701, 2018.

Limacher, E., Morton, C., Wood, D.: On the calculation of force from PIV data using the generalized added-mass and circulatory force decomposition. Experiments in Fluids, 60(1), 4, 2019.

Limacher, E., McClure, J., Yarusevych, S., Morton, C.: Comparison of momentum and impulse methods of force estimation using PIV data. 13th International Symposium on Particle Image Velocimetry (ISPIV), Munich, Germany, July 22-24, 2019.

Madsen, H. A., Bak, C., Døssing, M., Mikkelsen, R., Øye, S.: Validation and modification of the blade element momentum theory based on comparisons with actuator disc simulations. Wind Energy, 13(4), 373-389, 2010.

Noca, F.: PhD thesis, Caltech, 1997, https://thesis.library.caltech.edu/3081/.

Noca, F., Shiels, D., Jeon, D.: Measuring instantaneous fluid dynamic forces on bodies, using only velocity fields and their derivatives. Journal of Fluids and Structures, 11(3), 345-350, 1997.

Okulov, V. L., Sørensen, J. N., Wood, D. H. The rotor theories by Professor Joukowsky: Vortex theories. Progress in Aerospace Sciences, 73, 19-46 (2015).

Okulov, V. L., Sørensen, J. N. : Refined Betz limit for rotors with a finite number of blades. Wind Energy, 11, 415-426, 2008.

Rival, D. E., van Oudheusden, B.: Load-estimation techniques for unsteady incompressible flows Experiments in Fluids, 58, 20, (2017).

Saffman, P.G.: Vortex Dynamics, C.U.P. (1992).

https://doi.org/10.5194/wes-2019-93
Preprint. Discussion started: 19 December 2019

Sørensen, J. N.: General Momentum Theory for Horizontal Axis Wind Turbines, Springer International Publishing, Heidelberg, doi:10.1007/978-3-319-22114-4, 2016.

Taylor, G.I.: The "Rotational Inflow Factor" in Propeller Theory, ARC R & M no. 765 (Reprinted in G. Batchelor (ed) The Scientific Papers of G.I. Taylor, 3; 59 – 65, CUP).

Thomson, J.J.: A Treatise on the Motion of Vortex Rings, Macmillan, 1833.

Wood, D. H.: Small wind turbines: analysis, design, and application, Springer, 2011.vortical

Wu, J. C. Theory for aerodynamic force and moment in viscous flows. AIAA Journal, 19, 432–441, 1981.

Wu, J. Z., Ma, H. Y., Zhou, M. D.: Vortical Flows, Springer, Berlin, Heidelberg, 2015.