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A symbolic framework for flexible multibody systems applied to horizontal axis wind turbines

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Abstract. The article presents a symbolic framework that is used to obtain the linear and non-linear equations of motion of a multibody system including rigid and flexible bodies. Our approach is based on Kane's method and a nonlinear shape function representation for flexible bodies. The method yields compact symbolic equations of motion with implicit account of the constraints. The general and automatic framework facilitate the creation and manipulation of models with various levels of fidelity. The symbolic treatment allows for the obtention of analytical gradients and linearized equations of motion. The linear and non-linear equations can be exported to Python code or dedicated software. The application are multiple such as: time-domain simulation, stability analyses, frequency domain analyses, advanced controller design, state observers, digital twins, etc. In this paper, we describe the method we used to systematically generate the equations of motion of multibody systems. We apply the framework to generate illustrative onshore and offshore wind turbine models. We compare our results with OpenFAST simulations and discuss the advantages and limitations of the method. A Python implementation is provided as an opensource project.

1 Introduction

The next generation of wind turbine digital technologies requires versatile aero-servo-hydro-elastic models, with various levels of fidelity, suitable for a wide range of applications. Such applications include: time domain simulations, linearization (for controller design and tuning, or frequency domain analyses), analytical gradients (for optimization procedures), generation of dedicated, high-performance or embedded code (for standalone simulations, state observers or digital twins). Current models are implemented for a specific purpose and usually based on an heuristic structure. Aeroelastic tools, such as Flex (Øye, 1983; Branlard, 2019) or ElastoDyn (OpenFAST, 2021), rely on: an assumed chain of connections between bodies, a given set of degrees of freedom, and predefined orientations of shape functions. Tools with linearization capabilities, such as hawc-stab2 (Sønderby and Hansen, 2014) or OpenFAST (OpenFAST, 2021) are dedicated to horizontal axis wind turbines and the evaluation of the gradients are limited to hard-coded analytical expressions or numerical finite-differences. Small implementation changes often require an extensive redevelopment, and the range of applications of the tools remain limited (Simani, 2015).

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To address this issue, we propose a framework for the automatic derivation, processing and parametrization of models with granularity in the level of fidelity. Our approach is based on Kane's method (Kane and Wang, 1965) and a nonlinear shape function representation of flexible bodies (Shabana, 2013) described using a standard input data (SID) format (Wallrapp, 1994; Schwertassek and Wallrapp, 1999). The method yields compact symbolic equations of motion with implicit account of the constraints. Similar approaches have been presented in the literature: Kurz and Eberhard (2009), Merz (2018), Lemmer (2018), Branlard (2019). Our framework differs in the fact that all equations are processed at a symbolic level and therefore the model can be used in its nonlinear or linearized form. We implemented an open-source version in Python using SymPy (Sympy), leveraging its mechanical toolbox. Alternative symbolic frameworks found in the literature are usually limited to rigid bodies (Verlinden et al., 2005; Kurz and Eberhard, 2009; Gede et al., 2013; Docquier et al., 2013), or closed-source (Reckdahl and Mitiguy, 1996; Kurtz et al., 2010; MotionGenesis, 2016), and cannot be directly processed in Python.

In section 2, we present the formalism used to derive the equations of motion. In section 3, we given an overview of how the equations were implemented into a symbolic calculation framework, to easily manipulate the equations and generate dedicated code. Example of applications relevant to wind energy are given in section 4. Discussions and conclusions follow.

2 Method to obtain the equations of motion

In this section, we present the formalism used to setup the equations of motion.

2.1 System definition and kinematics

We consider a system of n_b bodies, rigid or flexible, connected by a set of joints. For simplicity, we assume that no kinematic loops are present in the system, and the mass of the bodies are constant. An inertial frame is defined to express the positions, velocities and accelerations of the bodies. We adopt a minimal set of generalized coordinates, $\boldsymbol{q}$, of dimension n_q, to describe the kinematics of the bodies: joint coordinates describing the joints displacements, and Rayleigh-Ritz coordinates for the amplitudes of the shape functions of the flexible bodies (see, e.g. Branlard (2019)). The choice of coordinates is left to the user, but it is assumed to form a minimal set. We will provide illustrative examples in section 4.

At a given time, the positions, orientations, velocities, and accelerations of all the points of the structure are entirely determined by the knowledge of $\boldsymbol{q}, \dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$. For a given body i, and a point P belonging to the body, the position, velocity and acceleration of the point are given by (see e.g. Shabana (2013)):

$$\boldsymbol{r}_P = \boldsymbol{r}_i + \boldsymbol{s}_P = \boldsymbol{r}_i + \boldsymbol{s}_{P_0} + \boldsymbol{u}_P \tag{1}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_i + \boldsymbol{\omega}_i \times \boldsymbol{s}_P + (\dot{\boldsymbol{u}}_P)_i \tag{2}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{s}_P) + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{s}_P + 2\boldsymbol{\omega}_i \times (\dot{\boldsymbol{u}}_P)_i + (\ddot{\boldsymbol{u}}_P)_i \tag{3}$$

where: $\boldsymbol{r}_i$, $\boldsymbol{v}_i$ and $\boldsymbol{a}_i$ are the position, velocity and acceleration of the origin of the body; $\boldsymbol{s}_{P_0}$ is the initial (undeformed) position vector of point P with respect to the body origin; $\boldsymbol{u}_P$ is the elastic displacement of the point (0 for rigid bodies); $\boldsymbol{\omega}_i$ is the rotational velocity of the body with respect to the inertial frame; $\dot{(\,)}$ and $\dot{(\,)}_i$ refer to time derivatives in the inertial and body frame respectively. Throughout the paper, we use bold symbols for vectors and matrices, and uppercase symbols for most matrices. The elastic displacement is obtained as a superposition of elastic deformations (see subsection 2.4). We define the transformation matrix $\boldsymbol{R}_i$ which transforms coordinates from the body frame to the inertial frame, and by definition $[\tilde{\boldsymbol{\omega}}_i] = \dot{\boldsymbol{R}}_i \boldsymbol{R}_i^T$, where $[\tilde{\ }]$ represents the skew symmetric matrix.

2.2 Introduction to Kane's method

Kane's method (Kane and Wang, 1965) is a powerful and systematic way to obtain the equations of motion of a system. The procedure leads to n_q coupled equations of motion:

$$\mathrm{f}_r + \mathrm{f}_r^* = 0, \qquad r = 1 \ldots n_q \tag{4}$$

where f_r^* is associated with inertial loads and f_r is associated with external loads, and these components are obtained for each generalized coordinates. The components are obtained as a superposition of contributions from each body:

$$\mathrm{f}_r = \sum_{i=1}^{n_b} \mathrm{f}_{ri}, \qquad \mathrm{f}_r^* = \sum_{i=1}^{n_b} \mathrm{f}_{ri}^* \tag{5}$$

The terms f_{ri} and f_{ri}^* can be obtained for each body individually and assembled at the end to form the final system of equations. We will present in subsection 2.3 and subsection 2.4 how these terms are defined for rigid bodies and flexible bodies respectively.

2.3 Rigid bodies

We assume that body i is a rigid body and proceed to define the terms f_{ri} and f_{ri}^*. The inertial force, $\boldsymbol{f}_i^*$, and inertial torque, $\boldsymbol{\tau}_i^*$, acting on the body are:

$$\boldsymbol{f}_i^* = -m_i \boldsymbol{a}_{G,i}, \qquad \boldsymbol{\tau}_i^* = -\boldsymbol{I}_{G,i} \cdot \dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times (\boldsymbol{I}_{G,i} \cdot \boldsymbol{\omega}_i) \tag{6}$$

where m_i is the mass of the body, $\boldsymbol{a}_{G,i}$ is the acceleration of its center of mass with respect to the inertial frame, $\boldsymbol{I}_{G,i}$ is the inertial tensor of the body expressed at its center of mass. Equation 6 is a vectorial relationship, it may therefore be evaluated in any coordinate system. The component f_{ri}^* is defined as:

$$\mathrm{f}_{ri}^* = \boldsymbol{J}_{v,ri} \cdot \boldsymbol{f}_i^* + \boldsymbol{J}_{\omega,ri} \cdot \boldsymbol{\tau}_i^* \tag{7}$$

with

$$\boldsymbol{J}_{v,ri} = \frac{\partial \boldsymbol{v}_{G,i}}{\partial \dot{q}_r}, \quad \boldsymbol{J}_{\omega,ri} = \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{q}_r} \tag{8}$$

where $\boldsymbol{v}_{G,i}$ is the velocity of the body masscenter with respect to the inertial frame. The partial velocities, or Jacobians, $\boldsymbol{J}_v$ and $\boldsymbol{J}_\omega$, are key variables of the Kane's method. They project the physical coordinates into the generalized coordinates ($\boldsymbol{q}$), inherently accounting for the kinematic constraints between bodies. In numerical implementations, the Jacobians are typically stored in matricial forms, referred to as velocity transformation matrices. The terms f^*_{ri} can equivalently be obtained using the partial velocity of any body point (e.g. the origin) by carefully transferring the inertial loads to the chosen point. The external forces and torques acting on the body are combined into an equivalent force and torque acting at the center of mass, written $\boldsymbol{f}_i$ and $\boldsymbol{\tau}_i$. The component f_{ri} is then given by:

$$\mathrm{f}_{ri} = \boldsymbol{J}_{v,ri} \cdot \boldsymbol{f}_i + \boldsymbol{J}_{\omega,ri} \cdot \boldsymbol{\tau}_i \tag{9}$$

Equivalently, the contributions from each individual force, $\boldsymbol{f}_{i,j}$, acting on a point P_j of the body i, and each individual torques, $\boldsymbol{\tau}_{i,k}$, can be summed using the appropriate partial velocity to obtain f_{ri}:

$$\mathrm{f}_{ri} = \sum_j \frac{\partial \boldsymbol{v}_{P_j}}{\partial \dot{q}_r} \cdot \boldsymbol{f}_{i,j} + \sum_k \boldsymbol{J}_{\omega,ri} \cdot \boldsymbol{\tau}_{i,k} \tag{10}$$

where $\boldsymbol{v}_{P_j}$ is the velocity of the point j with respect to the inertial frame. Equation 7 and Equation 9 are inserted in Equation 5 to obtain the final equations of motion.

2.4 Flexible bodies

We assume that body i is a flexible body and proceed to define the terms f_{ri} and f^*_{ri}. The dynamics of a flexible body is described in standards textbooks such as Shabana (2013) or Schwertassek and Wallrapp (1999). Unlike rigid bodies, the equations for flexible bodies are typically expressed with respect to a reference point different from the center of mass. We will call this point the origin and write it O_i.The elastic displacement field of the body is written $\boldsymbol{u}$. It defines the displacement of any point of the body with respect to its undeformed position. Using the first order[1] Rayleigh-Ritz approximation, the displacement field at a given point, P, is given by the sum of shape function contributions: $\boldsymbol{u}(P) = \sum_{j=1}^{n_{e,i}} \boldsymbol{\Phi}_{ij}(P)\boldsymbol{q}_{e,ij}(t)$, where $\boldsymbol{\Phi}_{ij}$ are the shape functions (displacement fields) of body i and $\boldsymbol{q}_{e,ij}$ is the subset of $\boldsymbol{q}$ consisting of the elastic coordinates of body i, of size $n_{e,i}$. The principles of the shape function approach applied to beams are given in Appendix B. The shape functions are more easily represented in the body coordinate system. Vectors and matrices that are explicitly written in the body frame will be written with primes. The equations of motion of the flexible bodies are (Wallrapp, 1994):

$$\begin{bmatrix} \boldsymbol{M}'_{xx} & \boldsymbol{M}'_{x\theta} & \boldsymbol{M}'_{xe} \\ & \boldsymbol{M}'_{\theta\theta} & \boldsymbol{M}'_{\theta e} \\ \text{sym.} & & \boldsymbol{M}'_{ee} \end{bmatrix}_i \begin{bmatrix} \boldsymbol{a}'_i \\ \dot{\boldsymbol{\omega}}_i \\ \ddot{\boldsymbol{q}}_{e,i} \end{bmatrix} + \begin{bmatrix} \boldsymbol{k}'_{\omega,x} \\ \boldsymbol{k}'_{\omega,\theta} \\ \boldsymbol{k}'_{\omega,e} \end{bmatrix}_i + \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{k}_e \end{bmatrix}_i = \begin{bmatrix} \boldsymbol{f}'_x \\ \boldsymbol{f}'_\theta \\ \boldsymbol{f}_e \end{bmatrix}_i \tag{11}$$

with: x, θ, e, subscripts that indicate the translation, rotation and elastic components; $\boldsymbol{M}$, the mass matrix of dimension $6+n_{e,i}$ made of the block matrices $\boldsymbol{M}_{xx}, \cdots, \boldsymbol{M}_{ee}$; $\boldsymbol{a}_i$ and $\dot{\boldsymbol{\omega}}_i$, the linear and angular acceleration of the body (origin) with respect to the inertial frame ; $\boldsymbol{k}_\omega$, the centrifugal, gyration and Coriolis loads, also called quadratic velocities loads; $\boldsymbol{k}_e$, the elastic strain loads ; $\boldsymbol{f}$, the external forces, torques and elastic generalized forces. The different components of $\boldsymbol{M}$, $\boldsymbol{k}_\omega$, $\boldsymbol{k}_e$ and $\boldsymbol{f}$ are given

[1]We address the second order approximation in subsection 5.3 and Appendix C.

in Appendix A. These terms depend on $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and $\boldsymbol{\Phi}_i$. The inertial force, torque and elastic loads are:

$$\boldsymbol{f}_i^* = -\boldsymbol{R}_i\left[\boldsymbol{M}'_{xx}\boldsymbol{a}'_i + \boldsymbol{M}'_{x\theta}\dot{\boldsymbol{\omega}}'_i + \boldsymbol{M}_{xe}\ddot{\boldsymbol{q}}_{e,i} + \boldsymbol{k}'_{\omega,x}\right] \tag{12}$$

$$\boldsymbol{\tau}_i^* = -\boldsymbol{R}_i\left[\boldsymbol{M}'_{\theta x}\boldsymbol{a}'_i + \boldsymbol{M}'_{\theta\theta}\dot{\boldsymbol{\omega}}'_i + \boldsymbol{M}_{\theta e}\ddot{\boldsymbol{q}}_{e,i} + \boldsymbol{k}'_{\omega,\theta}\right] \tag{13}$$

$$\boldsymbol{h}_i^* = - \quad \left[\boldsymbol{M}'_{ex}\boldsymbol{a}'_i + \boldsymbol{M}'_{e\theta}\dot{\boldsymbol{\omega}}'_i + \boldsymbol{M}_{ee}\ddot{\boldsymbol{q}}_{e,i} + \boldsymbol{k}'_{\omega,e}\right] \tag{14}$$

The external and elastic loads are:

$$\boldsymbol{f}_i = \boldsymbol{R}_i\boldsymbol{f}'_x \tag{15}$$

$$\boldsymbol{\tau}_i = \boldsymbol{R}_i\boldsymbol{f}'_\theta \tag{16}$$

$$\boldsymbol{h}_i = \boldsymbol{f}_e - \boldsymbol{k}_e \tag{17}$$

The component of f^*_{ri} and f_{ri}, for $r = 1\cdots n_q$, are then defined as:

$$\mathrm{f}^*_{ri} = \boldsymbol{J}_{v,ri}\cdot\boldsymbol{f}_i^* + \boldsymbol{J}_{\omega,ri}\cdot\boldsymbol{\tau}_i^* + \boldsymbol{J}_{e,ri}\cdot\boldsymbol{h}_i^* \tag{18}$$

$$\mathrm{f}_{ri} = \boldsymbol{J}_{v,ri}\cdot\boldsymbol{f}_i + \boldsymbol{J}_{\omega,ri}\cdot\boldsymbol{\tau}_i + \boldsymbol{J}_{e,ri}\cdot\boldsymbol{h}_i \tag{19}$$

with

$$\boldsymbol{J}_{v,ri} = \frac{\partial \boldsymbol{v}_{O,i}}{\partial \dot{q}_r}, \quad \boldsymbol{J}_{\omega,ri} = \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{q}_r}, \quad \boldsymbol{J}_{e,ri} = \frac{\partial \boldsymbol{q}_{e,i}}{\partial q_r} \tag{20}$$

where $\boldsymbol{v}_{O,i}$ is the velocity of the body with respect to the inertial frame. The term $\boldsymbol{J}_{e,ri}$ consists of 0 and 1 because $\boldsymbol{q}_{e,i}$ is a subset of $\boldsymbol{q}$. Equation 18 and Equation 19, once evaluated for body i, are inserted in Equation 5 to obtain the final equations of motion.

2.5 Non-linear and linear equations of motion

The n_q equations of motion given in Equation 4 are gathered into a vertical vector $\mathbf{e}$. They are recast into the form:

$$\mathbf{e}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{u},t) = \mathbf{f} + \mathbf{f}^* = \boldsymbol{F}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{u},t) - \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{0} \tag{21}$$

or

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{u},t) \tag{22}$$

where $\boldsymbol{M} = -\frac{\partial \mathbf{e}}{\partial \ddot{\boldsymbol{q}}}$ is the system mass matrix, and $\boldsymbol{F}$ is the forcing term vector, that is, the reminder terms of the equation ($\boldsymbol{F} = \mathbf{e} + \boldsymbol{M}\ddot{\boldsymbol{q}}$). The vector $\boldsymbol{u}$ is introduced to represent the time dependent inputs that are involved in the determination of the external loads. Both sides of the equations are also dependent on some parameters but this dependency is omitted to shorten notations. The stiffness and damping matrices may be obtained by computing the Jacobian of the equations of motion with respect to $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ respectively. The non-linear equation given in Equation 22 is easily integrated numerically, for instance by recasting the system into a first order system, or by using dedicated second order system time integrator.

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In various applications, a linear time invariant approximation of the system is desired. Such approximation is obtained at an operating point, noted with the subscript 0, which is a solution of the non-linear equations of motion, viz:

$$\mathbf{e}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \ddot{\boldsymbol{q}}_0, \boldsymbol{u}_0, t) = \mathbf{0} \tag{23}$$

The linearized equations about this operating point are obtained using a Taylor-Series expansion:

$$\boldsymbol{M}_0(\boldsymbol{q}_0)\boldsymbol{\delta}\ddot{\boldsymbol{q}} + \boldsymbol{C}_0(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \boldsymbol{u}_0)\boldsymbol{\delta}\dot{\boldsymbol{q}} + \boldsymbol{K}_0(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \boldsymbol{u}_0)\boldsymbol{\delta q} = \boldsymbol{Q}_0(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \boldsymbol{u}_0)\boldsymbol{\delta u} \tag{24}$$

with

$$\boldsymbol{M}_0 = -\left.\frac{\partial \mathbf{e}}{\partial \ddot{\boldsymbol{q}}}\right|_0, \quad \boldsymbol{C}_0 = -\left.\frac{\partial \mathbf{e}}{\partial \dot{\boldsymbol{q}}}\right|_0, \quad \boldsymbol{K}_0 = -\left.\frac{\partial \mathbf{e}}{\partial \boldsymbol{q}}\right|_0, \quad \boldsymbol{Q}_0 = \left.\frac{\partial \mathbf{e}}{\partial \boldsymbol{u}}\right|_0 \tag{25}$$

where $\boldsymbol{M}_0$, $\boldsymbol{C}_0$, $\boldsymbol{K}_0$ are the linear mass, damping, and stiffness matrices, $\boldsymbol{Q}_0$ is the linear forcing vector, δ indicate a small perturbation of the quantities, and $|_0$ indicates that the expressions are evaluated at the operating point. Examples of application of the linear equations of motion are: controller design, frequency domain analyses, stability analyses.

3 Implementation into a symbolic framework

In this section we discuss a Python open-source symbolic calculation framework that implements the equations given in section 2. A Maxima implementation from the same authors is also available Geisler (2021).

The python library YAMS (Yet Another Multibody Solver) started as a numerical tool published in previous work (Branlard, 2019). The library is now supplemented with a symbolic module so that both numerical and symbolic calculations can be achieved. The new implementation uses the python symbolic calculation package SymPy (Sympy). We leveraged the features present in the subpackage "mechanics", which contains all the tools necessary to compute kinematics: definition of frames and points, and determination of positions, velocities, and accelerations. The subpackage also contains an implementation of Kane's equations for rigid body (i.e. subsection 2.3). We were also inspired by the package PyDy (Gede et al., 2013), which is a convenient tool to export the equations of motion to executable code, and directly visualize the bodies in 3D. The core of our work consisted in implementing a class to define flexible bodies (`FlexibleBody`) and the corresponding Kane's method for this class (subsection 2.4).

For the `FlexibleBody` class, we followed the formalism of Wallrapp (1994), and implemented Taylor expansions for all the terms defined in Appendix A, allowing the symbolic computation with shape functions of any order. The different Taylor coefficients may be kept as symbolic terms, or replaced early on by numerical values provided for instance by an SID.

We structured the code into three layers. 1) The low-level layer integrates seamlessly with SymPy and PyDy, by using the `FlexibleBody` class we provide. It is the layer which offers the highest level of granularity and control for the user, since arbitrary systems with various kinematic constraints can be implemented, at the cost of requiring more expertise. 2) The second-level automates the calculation of the kinematics, by introducing simple connections between rigid and flexible bodies. The connections may be rigid, with constant offsets and rotations, or dynamic. A connection from a flexible body to another

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body is assumed to occur at one extremity of the flexible body. Some knowledge of SymPy mechanics are still required to use this layer. 3) The third level consists of template models such as generic onshore or offshore wind turbine models. Degrees of freedom are easily turned on and off for these models depending on the level of fidelity asked by the user and generic external forces can be implemented or declared as external inputs. The non-linear and linear equations of motion can be exported to latex and python-ready scripts for various applications (see subsection 5.1). As little as three lines of code are required by the user to perform the full step from derivation of the equations, optional linearization, and exportation. To obtain numerical results from the exported python code, the user needs to provide the arrays with the degrees of freedom values $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, their initial conditions,a dictionary with inputs ($\boldsymbol{u}$) that are function of time, and a dictionary of parameters ($\boldsymbol{p}$) containing all the numerical constants such as mass, acceleration of gravity, etc. We provide tools to readily time-integrate the generated model using numerical values (including initial values) from a set of OpenFAST input files.

The source code of YAMS is available on github as a subpackage of the Wind Energy LIBrary, WELIB (Branlard, 2021). The repository contains tests and working examples, including the ones presented in section 4. The finite element package of the repository can be used to generate the SID from beam models such as the ones used for blades and tower.

4 Wind energy applications

4.1 Approach

In this section we present different wind energy applications of the symbolic framework. We focus on models with at least one flexible body since the rigid-body formulation of SymPy has been well verified (Gede et al., 2013). For each example, the equations of motion are given and their results are compared with OpenFAST (OpenFAST, 2021) simulations. This is readily achieved because our framework can export the equations of motion to python functions, load input files from an OpenFAST model, and integrate the generated equations using the same conditions as defined in the OpenFAST input files. In this article, we do not focus on the modelling of the external loads but we include them in the equations of motion. It is the responsibility of the user to define these functions, for instance through aero- or hydro-force models. For the verification results presented in this section we only include the gravitational and inertial loading. In all examples, the NREL 5-MW reference wind turbine (Jonkman et al., 2009) is used. The examples below are provided on the github repository where the YAMS package is provided (Branlard, 2021).

4.2 Notations

We adopt a system of notations where the first letter of a body is used to identify the parameters of this body. As an example, the tower is represented with the letter T, and the following body parameters are defined: T, origin; M_T, mass; L_T, length; $(J_{x,T}, J_{y,T}, J_{z,T})$, diagonal coefficients of the inertia tensor about the center of gravity and in body-coordinates; $\boldsymbol{r}_{TG}$, vector from body origin to body center of mass, of coordinates (x_{TG}, y_{TG}, z_{TG}) in body-coordinates. We also define: θ_t, the nacelle tilt angle about the y axis; g the acceleration of gravity along $-z$; O the origin of the global coordinate system.

4.3 Rotating blade with centrifugal stiffening

We begin with the study of a flexible blade of length $L_B = R$, rotating at the constant rotational speed Ω. We use this test case to familiarize the reader with the key concepts of the shape function approach given in Appendix B. A sketch of the system is given in Figure 1. We start by modelling the blade using a single shape function, assumed to be directed along the x axis ("flapwise"): $\boldsymbol{\Phi}_1 = \Phi\hat{\boldsymbol{x}}$, where the hat notation indicates the unit vector in the x direction. The undeflected blade is directed

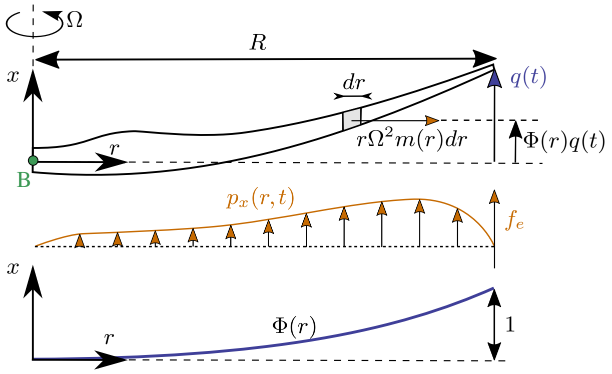

Figure 1. Sketch of a rotating blade, with the restoring centrifugal force. Points are indicated in green, degrees of freedom in blue, and loads in orange.

along the radial coordinate r, and rotates around the x axis. We assume that the shape function is known, noted $\Phi(r)$. It can be computed as the first flapwise mode of the blade, using tools provided in YAMS. The expression $\Phi(r) = r^3$ is a simple approximation that can be used for hand calculations. The aerodynamic force per length in the flapwise direction is noted $p_x(r)$. The generalized mass and stiffness are computed based on the mass per length (m) and flapwise bending stiffness (EI_y) of the blade, according to Equation B1:

$$M_e = \int_0^R m(r)\Phi^2(r)\,dr \tag{26}$$

$$K_e = \int_0^R EI_y(r)\left[\frac{d^2\Phi}{dr^2}(r)\right]^2\,dr \tag{27}$$

The generalized force is obtained from Equation B3:

$$f_e = \int_0^R p_x(r,t)\Phi(r)\,dr \tag{28}$$

The important consideration for this model is the axial load, N. The main axial load at a radial station r comes from the centrifugal force acting on all the points outboard of the current station:

$$N(r) = \int_r^R m(r')\Omega^2 r'\,dr' \tag{29}$$

The geometrical stiffness contribution of the axial load is obtained from Equation B5 as:

$$K_g(\Omega) = \int_0^R N(r)\left[\frac{d\Phi}{dr}\right]^2 dr = \Omega^2 \int_0^R \int_r^R m(r')r'\,dr' \left[\frac{d\Phi}{dr}\right]^2 dr \tag{30}$$

The axial stiffness, K_g, is positive and increases with the square of the rotational speed. This restoring effect is referred to as “centrifugal stiffening”. The natural frequency of the blade will increase with the rotational speed as follows:

$$\omega_0(\Omega) = \sqrt{\frac{(K_e + K_g(\Omega))}{M_e}} = \sqrt{\omega_0^2(0) + \frac{K_g(\Omega)}{M_e}} = \sqrt{\omega_0^2(0) + k_\Omega \Omega^2} \tag{31}$$

where k_Ω is referred to as the rise factor, or Southwell coefficient, and in our approximation, it is found to be constant: $k_\Omega = K_g(\Omega)/M_e/\Omega^2$. The coefficient provides the variation of the blade frequency with rotational speed, which is something that is observed on a Campbell diagram when performing stability analyses. In general, the mode shapes of the blade will also change as function of the rotational speed, and different shape functions should preferably be used for simulations at different rotational speed. The effect is fairly limited, and most OpenFAST practitioners only use one shape function corresponding to the value at rated rotational speed. Similarly, the Southwell coefficient is a function of the rotational speed, but the variations is negligible as long as the rotational speed is small compared to the natural frequency (e.g., $(\Omega/\omega)^2 \lesssim 5$, see Bielawa (2006)), which is the case for wind energy applications.

The treatment for a shape function in the edgewise direction is similar, using $\boldsymbol{\Phi}_2 = \Phi_2\hat{\boldsymbol{\theta}}$. In this case, the centrifugal force also has a component in the tangential direction equal to $p_{\theta,\text{centri}}(r) = -\Omega^2 u_\theta(r) dm(r)$, with $u_\theta = \Phi_2 q$. This leads to a generalized force equal to $\int_0^L p_{\theta,\text{centri}} dr \Phi_2 = -\Omega^2 M_e q$, or equivalently, to a stiffness term: $K_\omega = -\Omega^2 M_e$. It can be verified that this generalized force corresponds to the contribution $O_{e,11}\omega_x^2$, from $\boldsymbol{k}_{\omega,e}$, given in Equation A10. For an edgewise mode, the frequency therefore evolves as:

$$\omega_0(\Omega) = \sqrt{\frac{(K_e + K_g(\Omega) + K_\omega(\Omega))}{M_e}} = \sqrt{\omega_0^2(0) + (k_\Omega - 1)\Omega^2} \tag{32}$$

with $k_\Omega = K_g(\Omega)/M_e/\Omega^2$ and with K_g computed using Equation 30.

We apply the method to the NREL-5MW turbine using the blade properties and shape functions provided in the ElastoDyn input file. We order the degrees of freedom as: 1st flap, 1st edge and 2nd flap, assuming no coupling between the shape functions, so that they each of them can be treated individually using the results from this section. The diagonal coefficients of the mass matrix are $\text{diag}(\boldsymbol{M}_e) = [9.5e3,\ 1.5e4,\ 5.7e3]$, and for the stiffness matrix $\text{diag}(\boldsymbol{K}_e) = [1.7e4,\ 6.7e4,\ 8.7e4]$, computed according to Equation 26 and Equation 27. The coefficients k_Ω of each degrees of freedom are obtained as: $\boldsymbol{k}_\Omega = [1.7,\ 1.4,\ 5.5]$.

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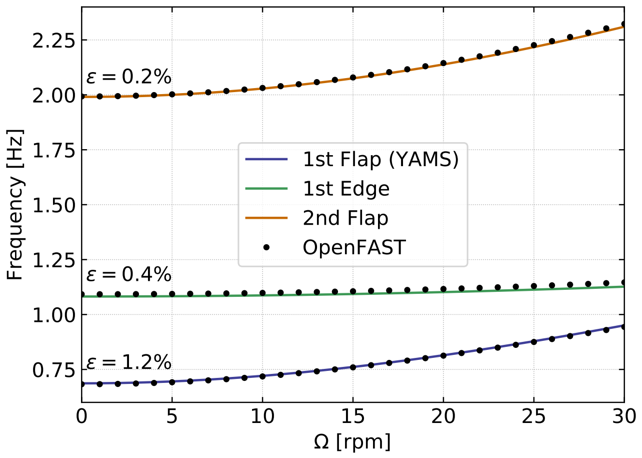

Figure 2. Variation of the natural frequencies of the NREL5-MW blade with rotational speed. Results from YAMS and OpenFAST, with mean relative error, ϵ, reported on the figure.

We compare the frequencies obtained with the present method against OpenFAST linearization results in Figure 2. The simulations were run in vacuum (no gravity, no aerodynamics) and with a cone angle of 0 deg. Strong agreement is found for the evolution of the different frequencies with the rotational speed. The stiffening is less pronounced for edgewise modes due to the softening introduced by K_ω.

This paragraph focused on the analysis of individual shape functions. In the general case, multiple shape functions are present and couplings might exists between them (due to the structural twist, non-orthogonality of the shape functions, or if the shape functions have components in multiple directions such as $\boldsymbol{\Phi_1} = \Phi_{1x}\hat{\boldsymbol{x}} + \Phi_{1y}\hat{\boldsymbol{y}}$). In such case, the general developments of Appendix A and Appendix B should be used.

4.4 Two degrees of freedom model of an onshore or fixed-bottom turbine

We consider a system of three bodies: tower (or support structure), nacelle and rotor. The system represents an onshore wind turbine or a fixed-bottom offshore wind turbine. A sketch of the system is given in Figure 3. The nacelle and rotor blades are rigid bodies, whereas the tower is flexible and represented by one shape function[2] in the fore-aft direction, noted $\boldsymbol{\Phi}_1 = \Phi_1\hat{\boldsymbol{x}}$. For hand calculations and as a first approximation, the first mode shape of a massless beam with a top-mass may be used: $\Phi_1(z) = 1 - \cos(z\pi/L/2)$. Increased accuracy is obtained when the shape function matches the actual first tower fore-aft bending mode accounting for the effect of the rotor-nacelle mass and inertia. The degrees of freedom are $\boldsymbol{q} = (q, \psi)$, where q is the generalized (elastic) coordinates in the fore-aft direction and ψ is the azimuthal position. The slope of the tower shape function at the tower top is a key coupling parameter of the model, noted ν_y. When the tower deflects $1\ \mathrm{m}$ in the x direction, the nacelle rotates by an angle ν_y. The methods assumes that the tower-top point remains along the x axis, neglecting the so-called non-linear geometrical effect. Yet, non-linear geometrical effects can be included using geometrical stiffening

[2] The relevant equations of the shape function approach for a beam are given in Appendix B.

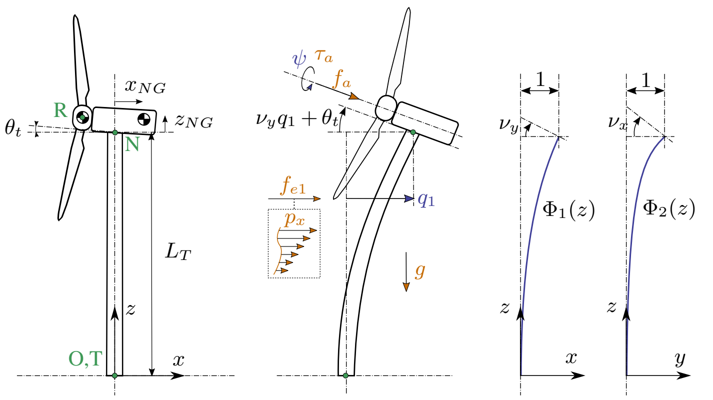

Figure 3. Model of an onshore or fixed-bottom wind turbine using 1 to 3 degrees of freedom (fore-aft and side-side flexibility of the support structure, and shaft rotation). Points are indicated in green, degrees of freedom in blue, and loads in orange.

corrections (Branlard, 2019). The aerodynamic thrust and torque are noted f_a, τ_a, acting at the rotor center (point R). The low-speed shaft generator torque is written τ_g. The distributed loads on the tower, p_x (from aero- and hydrodynamics), are projected against the shape function to obtain the generalized forces: $f_e = \int_0^{L_T} p_x(z,t)\Phi_1(z)dz$. The moments of inertia of the rotor in its coordinates are $(J_{x,R}, J_{\oplus,R}, J_{\oplus,R})$. We note M_e, K_e, D_e the generalized mass, stiffness and damping associated with a given shape function: $M_e = \int_0^{L_T} m(z)\Phi_1^2(z)dz$, $K_e = \int_0^{L_T} EI(z)\left[\frac{d^2\Phi_1}{dz^2}(z)\right]^2 dz$, $D_e = 2\zeta M_e \omega_e$ where $m(z)$ and $EI(z)$ are the mass per length and bending stiffness of the tower, and ω_e and ζ are the frequency and damping ratio associated with the shape function (assuming the shape function is close to a mode shape). The geometrical softening of the tower due to the tower top mass (K_{gt}) and its self-weight (K_{gs}) is obtained using Equation B5, as $K_g = K_{gt} + K_{gs}$, with :

$$K_{gt} = -g \int_0^{L_T} (M_R + M_N)\left[\frac{d\Phi_1}{dz}(z)\right]^2 dz \tag{33}$$

$$K_{gs} = -g \int_0^{L_T} \left[\frac{d\Phi_1}{dz}(z)\right]^2 \left[\int_z^{L_T} m(z')\,dz'\right] dz \tag{34}$$

The shape function frequency is obtained as:

$$\omega_e = \sqrt{(K_e + K_g)/M_e} \tag{35}$$

The application of the symbolic framework leads to the following equations of motion (rearranged for interpretability):

$$\begin{bmatrix} M_q & 0 \\ 0 & J_{x,R} \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} f_q \\ \tau_a - \tau_g \end{bmatrix} \tag{36}$$

where:

$$M_q = M_e + M_N + M_R \tag{37}$$
$$+ (J_{yN} + J_{\oplus,R} + M_N(x_{NG}^2 + z_{NG}^2) + M_R(x_{NR}^2 + z_{NR}^2))\nu_y^2 \tag{38}$$
$$+ 2\left[(M_N z_{NG} + M_R z_{NR})\cos(\nu_y q) - (M_N x_{NG} + M_R x_{NR})\sin(\nu_y q)\right]\nu_y \tag{39}$$

and

$$f_q = f_e - (K_e + K_g)q - D_e\dot{q} \tag{40}$$
$$+ g\nu_y\left[(M_N x_{NG} + M_R x_{NR})\cos(\nu_y q) + (M_N z_{NG} + M_R z_{NR})\sin(\nu_y q)\right] \tag{41}$$
$$+ \nu_y^2\dot{q}^2\left[(M_N x_{NG} + M_R x_{NR})\cos(\nu_y q) + (M_N z_{NG} + M_R z_{NR})\sin(\nu_y q)\right] \tag{42}$$
$$+ f_a\nu_y(x_{NR}\sin\theta_t + z_{NR}\cos\theta_t) \tag{43}$$
$$+ f_a\cos(\theta_t + \nu_y q) \tag{44}$$

Details on the derivations are given in Section D1. The mass matrix consists of three main contributions: Equation 37 represents the elastic mass and the rotor nacelle assembly (RNA) mass, Equation 38 is the generalized rotational inertia of the RNA, Equation 39 is the inertial coupling between the tower bending and the rotation of the nacelle. The forcing terms are identified as follows: Equation 40 consists of the elastic load due to the external forces on the tower, the elastic and geometric stiffness loads and the damping load on the tower; Equation 41 is the gravitational load from the RNA which will contribute to the stiffness of the system; Equation 42 is the centrifugal force of the RNA ("$M\omega^2 r$" with $\omega = \nu_y\dot{q}$); Equation 43 is the generalized torque from the aerodynamic thrust and Equation 44 is the thrust contribution acting directly along the direction of the shape function degree of freedom (along x). The RNA center of mass plays an important part in the equations (see the terms $(M_N x_{NG} + M_R x_{NR})$ and $(M_N z_{NG} + M_R z_{NR})$).

The equations of motion given in Equation 36 can be used to perform time domain simulations of a wind turbine. It is noted that the two degrees of freedom are only coupled by the aerodynamic loads. The non-linear model was used in previous work for time domain simulations and its linear version was used for state estimations (Branlard et al., 2020a, b). In this section, we apply the linearized form to compute the natural frequency of the turbine tower fore-aft mode. The linearized stiffness is here obtained by taking the gradient of the forcing with respect to q, and using a small angle approximation for ν_y to the second order:

$$K_{q,lin} = (K_e + K_g) - \nu_y^2(M_N g z_{NG} + M_R g z_{NR} - f_a q\cos\theta_t) + \nu_y f_a\sin\theta_t \tag{45}$$

For the NREL-5MW reference turbine (Jonkman et al., 2009), the different numerical values are: $g = 9.807$ m.s^{-2}, $\theta_t = 5$ deg, $x_{NR} = -5.0$ m, $z_{NR} = 2.4$ m, $L_T = 87.6$ m, $z_{NG} = 1.75$ m, $x_{NG} = 1.9$ m, $M_R = 1.1e5$ kg, $J_{x,R} = 3.86e7$ kg m^2, $J_{\oplus,R} = 1.92e7$ kg m^2, $M_N = 2.4e5$ kg, $J_{y,N} = 1.01e6$ kg m^2, $M_{RNA} = 3.5e5$ kg. The first fore-aft shape function of the

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NREL-5MW turbine tower, and its derivatives are:

$$\Phi_1(z) = (a_2\bar{z}^2 + a_3\bar{z}^3 + a_4\bar{z}^4 + a_5\bar{z}^5 + a_6\bar{z}^6)/(a_2+a_3+a_4+a_5+a_6)$$
$$\frac{d\Phi_1}{dz}(z) = \frac{1}{L_T}(2a_2\bar{z} + 3a_3\bar{z}^2 + 4a_4\bar{z}^3 + 5a_5\bar{z}^4 + 6a_6\bar{z}^5)/(a_2+a_3+a_4+a_5+a_6) \tag{46}$$
$$\frac{d^2\Phi_1}{dz^2}(z) = \frac{1}{L_T^2}(2a_2 + 6a_3\bar{z} + 12a_4\bar{z}^2 + 20a_5\bar{z}^3 + 30a_6\bar{z}^4)/(a_2+a_3+a_4+a_5+a_6)$$

with $\bar{z} = z/L$ and $a_2 = 0.7004$, $a_3 = 2.1963$, $a_4 = -5.6202$, $a_5 = 6.2275$, $a_6 = -2.504$. The material properties and the shape function are illustrated in Figure 4. The scaling of the shape functions given in Equation 46 is important to obtain the correct

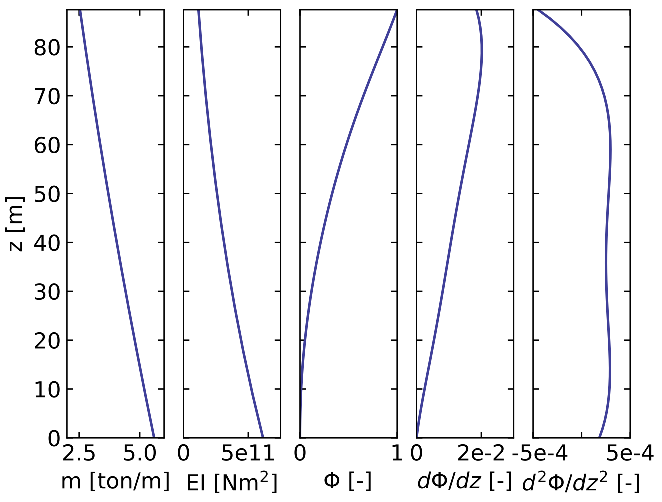

Figure 4. Properties of the NREL-5MW tower: mass per length (m), bending stiffness (EI), and shape function displacement (Φ), slope ($d\Phi/dz$) and curvature ($d^2\Phi/dz^2$).

numerical values for the flexible tower, namely: $\nu_y = 0.0185$, $M_e = 5.4e4$, $K_e = 1.91e6$, $K_g = -5.2e4 - 1.0e4 = -6.20e4$, $\omega_e = \sqrt{(K_e + K_g)/M_e} = 5.85$ rad/s. These numerical values, with $q = 0$, leads to: $M_q = 4.375e5$ and $K_q = 1.849e9$. The first fore-aft mode of the wind turbine has a natural frequency of $f = \sqrt{K_q/M_q} = 0.3272$ Hz. This value was compared with results obtained using OpenFAST linearization. Both methods are in strong agreements with differences only arising at the fifth decimal place.

4.5 Three degrees of freedom model of an onshore or fixed-bottom turbine

We consider the same system as the one presented in subsection 4.4 but the tower is now represented by one shape function in both the fore-aft and side-side directions, $\boldsymbol{\Phi}_1 = \Phi_1\hat{\boldsymbol{x}}$ and $\boldsymbol{\Phi}_2 = \Phi_2\hat{\boldsymbol{y}}$. The degrees of freedom are $\boldsymbol{q} = (q_1, q_2, \psi)$, where q_1 and q_2 are the generalized (elastic) coordinates in the fore-aft and side-side directions respectively, and ψ is the rotor azimuth. A sketch of the system is given in Figure 3.

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The slopes of the shape functions at the tower top are key coupling parameters of the model, noted ν_x and ν_y. The aerodynamic thrust and torque are noted f_a, τ_a, acting at point R. The distributed loads on the tower, p_x and p_y (from aero- and hydrodynamics), are projected against the shape functions to obtain the generalized forces $f_{e1} = \int \Phi_1 p_x dz$ and $f_{e2} = \int \Phi_2 p_y dz$. The moments of inertia of the rotor in its coordinates are $(J_{x,R}, J_{\oplus,R}, J_{\oplus,R})$. We note $\boldsymbol{M}_e, \boldsymbol{K}_e, \boldsymbol{D}_e$ the generalized mass, stiffness and damping associated with a given shape function (e.g. $M_{e11} = \int \Phi_1^2 m(z) dz$, with m the mass per length of the tower). The application of the symbolic framework leads to the equations of motion given in Section D2. To simplify the equations and limit their length when printing them in this article, we have applied a first-order small angle approximation for θ_t, and second order for ν_x and ν_y. It is seen from Equation D14, that a first order approximation for ν_y would have removed the influence of the rotor and nacelle y-inertia on the generalized mass associated with the tower fore-aft bending.

We performed a time simulation of the model using both our symbolic framework YAMS and OpenFAST. We compare the time series obtained using our generated functions with results from the equivalent OpenFAST simulation in Figure 5. In this simulation, the tower top is initially displaced by 1 m in the x and y directions, and the rotational speed is 5 rpm. We report the

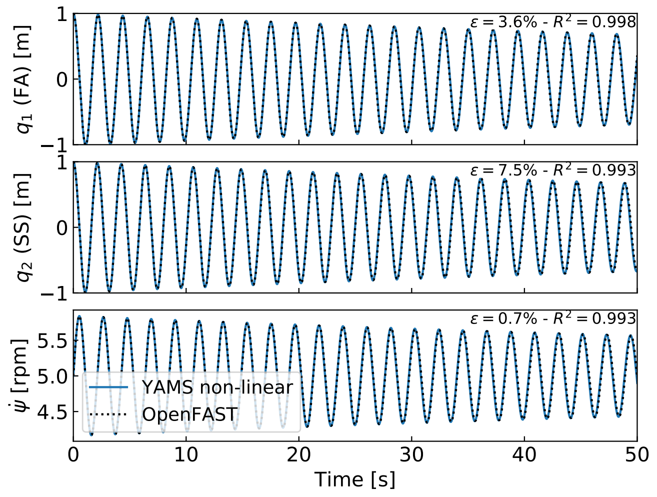

Figure 5. Free decay results for the onshore/fixed bottom model using both the symbolic framework (YAMS) and OpenFAST. From top to bottom: tower fore-aft bending, tower side side bending, and shaft rotational speed.

mean relative error, ϵ, and the coefficient of determination R^2 on the figure. We observe that our model is in strong agreement with the OpenFAST simulation. The differences in the second tower degree of freedom are attributed to: 1) the handling of the small angle approximation which is different in OpenFAST (using the closest orthonormal matrix, Jonkman (2009)) and in our formulation (two successive rotations, linearized); 2) the non-linear geometric corrections that are implemented in OpenFAST and which we have here omitted by only selecting shape function expansion to the first order (see subsection 5.3). The variation in azimuthal speed, resulting from the coupling between the gyroscopic loads and the tower bending, is well captured.

4.6 Three degrees of freedom model of a floating turbine

In this example, we demonstrate the applicability of the method for a floating wind turbine. We model the turbine using 3 bodies: rigid floater, flexible tower, rigid rotor-nacelle-assembly (labelled "N"). The degrees of freedom selected are: $\boldsymbol{q} = (x, \phi, q_T)$, where x is the floater surge, ϕ, the floater pitch, and q_T, the coordinate associated with a selected fore-aft shape function. A sketch of the model is given in Figure 6. The notations are similar to the ones presented in subsection 4.5. Lumped

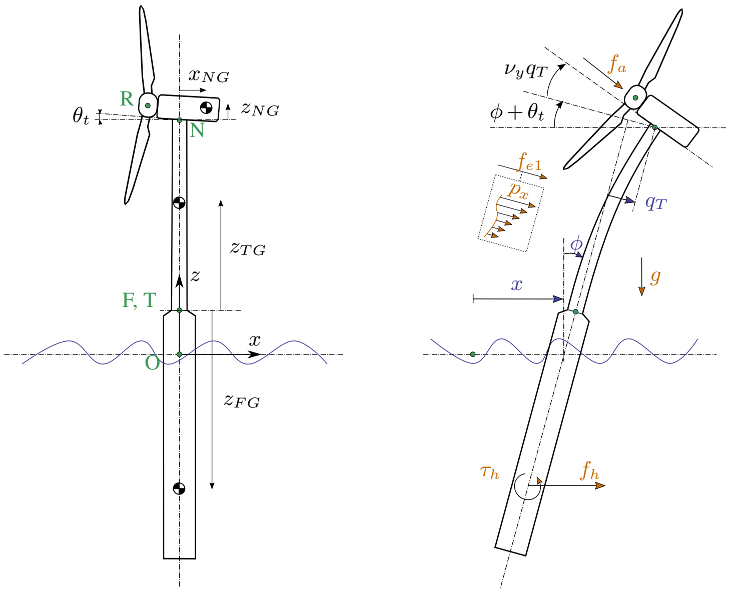

Figure 6. Model of an floating wind turbine using 3 degrees of freedom. Points are indicated in green, degrees of freedom in blue, and loads in orange.

hydrodynamic loads at the floater center of mass are now added. The model can also be used for a combined tower and floater that is flexible, simply by setting the mass of the floater to zero and including the hydrodynamic loading into the loading p_x. The equations of motion are given in subsection D3. The equations were simplified using a first order small angle approximation of θ_t and ϕ_y, and a second order approximation for ν_y.

We performed a numerical simulation of the model generated by YAMS and compared it with OpenFAST, for a case with gravitational loads only, starting with $x = 0$ m, $\phi = 2$ deg and $q_T = 1$ m. The results are presented in Figure 7. We observe again that the results from the two models correlate to a high degree.

We also compared the linearized version of both models. The symbolic framework can generate the linearized mass, stiffness and damping matrices as described in subsection 2.5. The matrices are then combined into a state matrix and compared with the state matrices written by the OpenFAST linearization feature. The eigenvalue analysis of the YAMS state matrix returned a pitch and fore-aft frequencies of 0.099 Hz and 0.799 Hz respectively, whereas OpenFAST returned 0.095 Hz and 0.795 Hz. The 4% error in the pitch frequency appears reasonable in view of the approximations used.

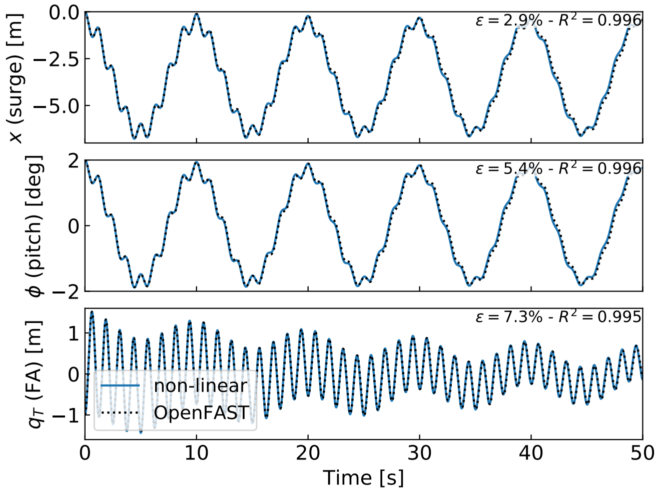

Figure 7. Free decay results for the floating wind turbine model using YAMS and OpenFAST. From top to bottom: surge, pitch and tower fore-aft bending.

5 Discussions

5.1 Applications

The implementation of the symbolic YAMS library was originally motivated by the need to obtain a simple linearized model of a floating wind turbine for frequency domain simulations. The potential applications of the framework are yet multiple:

- The generated equations can be used in time-domain simulation tools. The equations can be readily exported to different programming languages (C, Fortran or Python) providing computationally efficient tools, particularly because the method generates compact, and minimal equations. This is in contrast to most other multibody codes where many terms are calculated as matrix equations and through successive function calls. Further, the symbolic framework allows to generate optimized code, where common terms and factors are computed once and stored in temporary variables for reuse in the different expressions. In our examples, time-domain simulations were observed to be two orders of magnitude faster when using the automatically generated code in Python compared to OpenFAST simulations that rely on a compiled-language. Using such a framework can be considered in the future to replace the existing ElastoDyn module of OpenFAST. It can also be applied to unusual configurations such as multi-rotor or some vertical axis turbine concepts. Dedicated code can be generated for specific applications for increased performances. For instance, implicit integrator with iterative Newton-Raphson-like solvers benefit from the possibility to generate exact and efficient Jacobians along with the equations of motion.

- The generation of linearized models have a wide range of applications, such as: linear time-domain simulations, controller design and tuning, frequency domain analyses, stability analyses, state observers or digital twins. The symbolic approach is severalfold faster than alternative approaches because it can be evaluated for all operating points at once, whereas other methods (e.g. OpenFAST, HawcStab2) require multiple linearization calls. Linearization with respect to parameters can also be performed, making the method even more appealing, e.g. for controller designs based on linear parameter varying approaches such as linear matrix-inequality based designs (Pöschke et al., 2020).

– Analytical gradients of the equations can be computed and used in optimizations, nonlinear model predictive control or moving horizon estimation. External loads that cannot be expressed analytically can be defined as generic functions of the structural degrees of freedom, inputs, and parameters. After the code generation, the user can link a numerical implementation of the function and its numerical gradients to be able to use a mix of analytical and numerical gradients.

5.2 Advantage of using a symbolic framework

Most advantages have already been discussed in subsection 5.1, namely: the wide range of applications and the potential gain in computational time. Additionally, the method can provide useful insights and be used as an educational tool: simple models of a system with few degrees of freedom can readily be obtained, studied, and compared to hand-based calculation.

5.3 Advanced consideration

Section 2 addressed the systematic derivation of the equations of motion for an assembly of rigid or flexible bodies. Some advanced aspects of the method are discussed here:

– The expression of the displacement field $\boldsymbol{u}$ in terms of a superposition of shape function is typically done using a first or second order expansion. We discuss these formulations in Appendix C. Our framework supports both approaches: it is the responsibility of the user to provide the terms and values of the expansion when numerical evaluation is to occur. The advantage of the second-order expansion is that some geometrical non-linearities are directly accounted for by the method, whereas these non-linearities need to be introduced “manually” in the first-order expansion approach, as done in Branlard (2019).

– The definition of geometric stiffening requires attention in the general case. It is accounted for by the term $\boldsymbol{k}_\sigma$ presented in Appendix A. A general account of the effect for an arbitrary geometry can be found in Wallrapp and Schwertassek (1991) and simplified expressions are given for beams in Appendix B.

– The treatment of external loads was not addressed in details in this paper because the loads are application specific (aerodynamics, hydrodynamics, etc.). The framework can accept external loads as arbitrary functions of multiple variables, or as analytical expressions. In the former case, the user will have to provide an implementation of the function during the execution.

– Even though the equations of motion are void of constraint forces, the values of these forces can be recovered. They can be expressed as function of the external forces and the states of the system. It is not necessary to compute them by iteratively solving constraint equations.

- The framework can easy include rheonomous constraints, e.g. for the pitch angle, without having to supply a dedicated torque. Pitch speed and accelerations can be directly introduced into the mechanical system if they are provided e.g. by a generic second order pitch actuator model

5.4 Limitations

In spite of the advantages listed in subsection 5.2, the symbolic procedure presented in this work has some potential limitations. We are identifying two in this section. First, constraints and closed loops have currently not been added to the framework. The SymPy mechanics package supports additional constraint equations within the Kane method. It is therefore hoped that this current limitation can be lifted in the future. Second, large problems may challenge a symbolic calculation package: memory impact, calculation time, simplification times, and size of expressions may become significant. Some of these issues may be alleviated by introducing intermediate variables that are only substituted for in the numerical implementation or by using a recursive formulation of the solution procedure (Branlard, 2019).

6 Conclusions

We presented a symbolic framework to obtain the linear and non-linear equations of motion of a multibody system made of rigid bodies, flexible bodies and kinematic joints. Our approach is based on Kane's method and a nonlinear shape function representation of flexible bodies. We provided different wind energy examples and verified the results against OpenFAST simulations. The framework can readily provide models suitable to a wide range of applications, with competitive computational times. The framework is opensource and the examples presented are available in the repository. Future work will focus on applying the framework to dedicated research projects, with more complex systems, and potentially extend the framework to account for closed-loop systems and arbitrary constraints.

Author contributions. Both authors exchanged over the last two years on the implementation of such framework and its application to wind energy. E. Branlard wrote a Python implementation and J. Geisler wrote a Maxima implementation. E. Branlard wrote the main corpus of the paper, with feedback and contributions from J. Geisler.

Competing interests. No competing interests are present.

Appendix A: Equations for a flexible body and shape integrals

In this section, we detail the equations of motion of a flexible body. The reader is referred to the following references for a complete treatment of the equations of motion: Shabana (2013), Schwertassek and Wallrapp (1999) and Wallrapp (1994). The subscript i indicating the body index is dropped. The number of flexible shape functions associated with the body is n_e, the flexible degrees of freedom are $\boldsymbol{q}_e$, and the shape functions are gathered into a matrix $\boldsymbol{\Phi}$ of size $(3 \times n_e)$. Primes are used to indicate that quantities are expressed in the body frame of reference. The equations of motion, given in Equation 11, are repeated below:

$$\begin{bmatrix} \boldsymbol{M}'_{xx} & \boldsymbol{M}'_{x\theta} & \boldsymbol{M}'_{xe} \\ & \boldsymbol{M}'_{\theta\theta} & \boldsymbol{M}'_{\theta e} \\ \text{sym.} & & \boldsymbol{M}'_{ee} \end{bmatrix}_i \begin{bmatrix} \boldsymbol{a}'_i \\ \dot{\boldsymbol{\omega}}_i \\ \ddot{\boldsymbol{q}}_{e,i} \end{bmatrix} + \begin{bmatrix} \boldsymbol{k}'_{\omega,x} \\ \boldsymbol{k}'_{\omega,\theta} \\ \boldsymbol{k}_{\omega,e} \end{bmatrix}_i + \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{k}_e \end{bmatrix}_i = \begin{bmatrix} \boldsymbol{f}'_x \\ \boldsymbol{f}'_\theta \\ \boldsymbol{f}_e \end{bmatrix}_i \tag{A1}$$

The different terms of the mass matrix are obtained as follows:

$$\boldsymbol{M}'_{xx} = \int \boldsymbol{I}_3 \,\mathrm{d}m = M\boldsymbol{I}_3 \qquad (3 \times 3) \tag{A2}$$

$$\boldsymbol{M}'_{x\theta} = -\int \tilde{\boldsymbol{s}}'_P \,\mathrm{d}m = -M\tilde{\boldsymbol{s}}'_{CM} \qquad (3 \times 3) \tag{A3}$$

$$\boldsymbol{M}'_{\theta\theta} = -\int \tilde{\boldsymbol{s}}'_P \tilde{\boldsymbol{s}}'_P \,\mathrm{d}m = \boldsymbol{J} \qquad (3 \times 3) \tag{A4}$$

$$\boldsymbol{M}'_{\theta e} = \int \tilde{\boldsymbol{s}}'_P \boldsymbol{\Phi}' \,\mathrm{d}m = \boldsymbol{C}_r^T \qquad (3 \times n_e) \tag{A5}$$

$$\boldsymbol{M}'_{xe} = \int \boldsymbol{\Phi}' \,\mathrm{d}m = \boldsymbol{C}_t^T \qquad (3 \times n_e) \tag{A6}$$

$$\boldsymbol{M}'_{ee} = \int \boldsymbol{\Phi}'^T \boldsymbol{\Phi}' \,\mathrm{d}m \qquad (n_e \times n_e) \tag{A7}$$

The integrals are understood as volume integrals over the volume of the body (for beams they reduce to line integrals). The notation $[\tilde{\ }]$ represents the skew symmetric matrix. M is the mass of the body. The vector $\boldsymbol{s}'_{CM}$ is the vector from the origin of the body to undeflected center or mass (CM) of the body. The notations $\boldsymbol{C}_t$ $(n_e \times 3)$ and $\boldsymbol{C}_r$ $(n_e \times 3)$ are introduced to match Wallrapp's notations. The vector $\boldsymbol{s}'_P$ is the vector from the origin of the body to a deflected point of the body of elementary mass $\mathrm{d}m$. The undeflected position of this point is written $\boldsymbol{s}'_{P_0}$ and the displacement field $\boldsymbol{u}'$, such that: $\boldsymbol{s}'_P = \boldsymbol{s}'_{P_0} + \boldsymbol{u}'$. For a first order expansion of the displacement field, $\boldsymbol{u}' = \boldsymbol{\Phi}'\boldsymbol{q}_e$. Second order expansions need the introduction of an additional notation: $\boldsymbol{u}' = \boldsymbol{\Phi}'_u(\boldsymbol{q}_e)\boldsymbol{q}_e$ (see subsection 5.3 and Wallrapp (1994)). Wallrapp also includes the elementary mass moment of inertia which results in additional terms in the integrals (see Wallrapp (1994)). Such contributions are relevant for instance when considering the torsion of a beam (see Branlard (2019)). The block matrices $\boldsymbol{M}'_{xx}$, $\boldsymbol{M}'_{xe}$ and $\boldsymbol{M}'_{ee}$ do not depend on the deformation of the body and are hence constant. The other terms are functions of $\boldsymbol{q}_e$. They may be expressed as linear combination of constant integrals. These integrals are usually referred to as shape integrals (Shabana (2013)) or Taylor series coefficients (Wallrapp (1994)).

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The quadratic velocity terms $\boldsymbol{k}_\omega$ are given as:

$$\boldsymbol{k}'_{\omega,x} = 2\tilde{\boldsymbol{\omega}}'\boldsymbol{C}_t^T\dot{\boldsymbol{q}}_e + M\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}'\boldsymbol{s}'_{CM} \qquad (3\times 1) \tag{A8}$$

$$\boldsymbol{k}'_{\omega,\theta} = \tilde{\boldsymbol{\omega}}'\boldsymbol{M}'_{\theta\theta}\boldsymbol{\omega}' + \left[\sum_{j=1..n_e}\boldsymbol{G}_{r,j}\dot{q}_{e,j}\right]\boldsymbol{\omega}' \qquad (3\times 1) \tag{A9}$$

$$\boldsymbol{k}_{\omega,e} = \left[\boldsymbol{\omega}'^T\boldsymbol{O}_{e,j}\boldsymbol{\omega}'\right]_{j=1..n_e} + \left[\sum_{j=1..n_e}\boldsymbol{G}_{e,j}\dot{q}_{e,j}\right]\boldsymbol{\omega}' \qquad (n_e\times 1) \tag{A10}$$

where

$$\boldsymbol{G}_{r,j} = -2\int \tilde{\boldsymbol{s}}'_P\tilde{\boldsymbol{\Phi}}'_j\,\mathrm{d}m \qquad (3\times 3) \tag{A11}$$

$$\boldsymbol{O}_{e,j} = \int \tilde{\boldsymbol{\Phi}}'_j\tilde{\boldsymbol{s}}'_P\,\mathrm{d}m = -\frac{1}{2}\boldsymbol{G}_{r,j}^T \qquad (3\times 3) \tag{A12}$$

$$\boldsymbol{G}_{e,j} = -2\int \boldsymbol{\Phi}'^T\tilde{\boldsymbol{\Phi}}'_j\,\mathrm{d}m \qquad (n_e\times 3) \tag{A13}$$

The first term of Equation A10 is obtained by vertically stacking the contribution of each shape function. In the SID format, this term is reshaped as the product $\boldsymbol{O_e\Omega}$, where:

$$\boldsymbol{O}_e = [O_{e,j,11},\, O_{e,j,22},\, O_{e,j,33},\, O_{e,j,12}+O_{e,j,21},\, O_{e,j,23}+O_{e,j,32},\, O_{e,j,13}+O_{e,j,31}]_{j=1..n_e} \qquad (n_e\times 6) \tag{A14}$$

$$\boldsymbol{\Omega} = \left[\omega_x^2,\,\omega_y^2,\,\omega_z^2,\,\omega_x\omega_y,\,\omega_y\omega_z,\,\omega_x\omega_z\right]' \qquad (6\times 1) \tag{A15}$$

The body elastic forces are given by:

$$\boldsymbol{k}_e = \boldsymbol{k}_\sigma + \boldsymbol{K}_e\boldsymbol{q}_e + \boldsymbol{D}_e\dot{\boldsymbol{q}}_e \tag{A16}$$

where $\boldsymbol{K}_e$ and $\boldsymbol{D}_e$ are the elastic stiffness and damping matrix, and $\boldsymbol{k}_\sigma$ represent stress stiffening terms. The elastic damping forces are often given as stiffness proportional damping. For more details, see Wallrapp (1994), and for more example of with elastic beams see Branlard (2019). The external loads can be assumed to consists of distributed volume forces $\boldsymbol{p}'$ (in practice they are mostly surface forces of line forces) and a gravitational acceleration field $\boldsymbol{g}'$. The components of the external loads in Equation A1 are then obtained by integration over the whole body:

$$\boldsymbol{f}'_x = \int \boldsymbol{p}'\,dV + \boldsymbol{M}'_{xx}\boldsymbol{g}' \qquad (3\times 1) \tag{A17}$$

$$\boldsymbol{f}'_\theta = \int \boldsymbol{s}'_P\times\boldsymbol{p}'\,dV + \boldsymbol{M}'_{\theta x}\boldsymbol{g}' \qquad (3\times 1) \tag{A18}$$

$$\boldsymbol{f}'_e = \int \boldsymbol{\Phi}'^T\boldsymbol{p}'\,dV + \boldsymbol{M}'_{ex}\boldsymbol{g}' \qquad (n_e\times 1) \tag{A19}$$

Appendix B: Application of the shape function approach to an isolated beam

In this section, we illustrate how the elastic equations of Appendix A can be applied to an isolated beam. Examples of applications are further given in subsection 4.3 and subsection 4.4. We consider a beam directed along the z-axis and bending in the x and y direction. Expression are written in the coordinate system of the beam and primes are dropped in this section. The beam properties are the following: length L, mass per length m, and bending stiffness EI_x and EI_y. We assume that the displacement field is such that the shape functions are function of z only: $\boldsymbol{u}(z,t) = \sum_{i=1}^{n_e} \boldsymbol{\Phi}_i(z) q_{e,i}(t)$. We also assume that the shape functions satisfy at least the geometric boundary conditions. The kinetic energy of the beam is $T = \frac{1}{2}\int_0^L m\dot{u}^2 dz = \frac{1}{2}\sum_i\sum_j M_{e,ij}\dot{q}_{e,j}\dot{q}_{e,i}$. where $M_{e,ij}$ is (see Equation A7):

$$M_{e,ij} = \int_0^L m(z)\,\boldsymbol{\Phi}_i(z)\cdot\boldsymbol{\Phi}_j(z)\,dz, \quad i,j = 1,\dots n_e \tag{B1}$$

Equation B1 involves a scalar product of the shape functions at each spanwise positions. Integrals over the moment of inertia can be used to account for torsion (see Branlard (2019)). The potential energy (strain energy) of the beam, is obtained as $V = \frac{1}{2}\sum_i\sum_j K_{e,ij} q_{e,i} q_{e,j}$, where $K_{e,ij}$ are the element of the stiffness matrix, which, under the assumption of small deformations, are given by:

$$K_{e,ij} = \int_0^L \left[EI_y \frac{d^2\Phi_{i,x}}{dz^2}\frac{d^2\Phi_{j,x}}{dz^2} + EI_x \frac{d^2\Phi_{i,y}}{dz^2}\frac{d^2\Phi_{j,y}}{dz^2}\right] dz, \quad i,j = 1,\dots n_e \tag{B2}$$

Elongation and torsional strains (EA and GK_t) can similarly be added to the strain energy and the stiffness matrix if longitudinal and torsional displacement fields are included in the shape functions. The external loads on the beam are assumed to consist of a distributed force vector $\boldsymbol{p}(z)$. The virtual work done by the force $\boldsymbol{p}$ for each virtual displacement $\delta q_{e,i}$, provides the generalized force as(see Equation A17):

$$f_{e,i} = \int_0^L \boldsymbol{\Phi}_i \cdot \boldsymbol{p}\,dz \tag{B3}$$

The equations of motion of the isolated beam and then written in matricial form as:

$$\boldsymbol{M}_e\ddot{\boldsymbol{q}}_e + \boldsymbol{D}_e\dot{\boldsymbol{q}}_e + \boldsymbol{K}_e\boldsymbol{q}_e = \boldsymbol{f}_e \tag{B4}$$

where $\boldsymbol{q}_e = [q_{e,1},\cdots,q_{e,n}]$. Damping is typically added a posteriori to the equations, where the Rayleigh damping assumption is often used $\boldsymbol{D}_e = \alpha\boldsymbol{M}_e + \beta\boldsymbol{K}_e$ (stiffness proportional damping implies $\alpha = 0$). If the shape functions are mode shapes, then the shape functions are orthogonal, the mass and stiffness matrices are diagonal, and the stiffness values would be $K_{e,ii} = \omega_{e,i}^2 M_{e,ii}$, with $\omega_{e,i} = \sqrt{K_{e,ii}/M_{e,ii}}$ the eigenfrequency of the beam mode i. The modal damping is then given by $D_{e,ii} = 2\zeta_i M_{e,ii}\omega_{e,i}$, where ζ_i is the damping ratio associated with mode i.

If the beam is loaded axially by a force $N(z)$, then this force produces a distributed load in the transverse direction equal to $\boldsymbol{n} = \frac{\partial}{\partial z}\left[N(z)\frac{\partial \boldsymbol{u}}{\partial z}\right]$, with components in the y and z directions (see Branlard (2019)). The generalized force associated with this loading is then: $Q_{N,i} = \int_0^L \boldsymbol{\Phi}_i \cdot \boldsymbol{n}\,dz$. Inserting the expression of $\boldsymbol{n}$ and $\boldsymbol{u}$, the generalized force has the form of a stiffness term: $Q_{N,i} = -\sum_j K_{N,ij} q_j$ with

$$K_{N,ij} = -\int_0^L \boldsymbol{\Phi}_i \cdot \frac{d}{dz}\left[N(z)\frac{d\boldsymbol{\Phi}_j}{dz}\right] dz = \int_0^L N(z)\frac{d\boldsymbol{\Phi}_i}{dz}\cdot\frac{d\boldsymbol{\Phi}_j}{dz} - \left[N(x)\boldsymbol{\Phi}_i \cdot \frac{d\boldsymbol{\Phi}_j}{dz}\right]_0^L \tag{B5}$$

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and where integration by parts was used to obtain the second equality. Examples of applications are given in subsection 4.3 and subsection 4.4.

Appendix C: Alternative formulations

Different formulations of flexible multibody dynamics are found in the literature. Some of the alternatives are briefly discussed in this section.

In Equation 7, the Jacobian terms $\boldsymbol{J}$ and the virtual work are expressed in vector form. In such form, there is no need to precise in which coordinate system the different vectors are expressed. This is convenient to reduce the size of the expressions when using symbolic calculations. In a numerical framework, the vector will have to be expressed in a common frame. When such approach is used (see e.g. Lemmer (2018); Branlard (2019)), the Jacobians are sometimes stacked into a matricial form:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_v \\ \boldsymbol{J}_\omega \\ \boldsymbol{J}_e \end{bmatrix} \tag{C1}$$

Some implementation choices are needed depending if these matrices are expressed in the global frame or a body frame. The Jacobian matrices are referred to as velocity transformation matrix, and the link between formulations in global and local coordinates is given in Branlard (2019). In the same reference, recursive relationships are given for tree-like assembly of bodies, to help express the Jacobian matrices of each body recursively, based on the matrices of the parent body. It is also noted that the quadratic velocity terms, $\boldsymbol{k}_\omega$, can be obtained using the time derivative of the Jacobian matrix.

In this article, we have not explicitly written the rotational impact of the shape functions. In most applications, bodies are connected at their extremities and the deflection slope at a body extremity will induce a rotation of the subsequent body. The deflection slope can be obtained form the knowledge of the shape functions. This is readily accounted for by introducing time-varying rotation matrix between bodies, and this is the approach used in our symbolic framework. A formalism of rotations of bodies connected at their extremities is given in Branlard (2019). A more general formulation, introducing shape function rotations $\boldsymbol{\Psi}$, is given in (Wallrapp, 1994; Schwertassek and Wallrapp, 1999; Lemmer, 2018). In such formulation, the linear rotation field is obtained as $\boldsymbol{I} + \widetilde{\boldsymbol{\Psi q}}$, where $\boldsymbol{I}$ is the identity matrix.

The order of expansion of the displacement field leads to alternative formulations. In Shabana (2013) and Branlard (2019) a first order expansion is used: $\boldsymbol{u} = \sum_j [\boldsymbol{\Phi}_j^0] q_{e,j}$ In the work of Wallrapp a second order expansion is used: $\boldsymbol{u} = \sum_j \left[\boldsymbol{\Phi}_j^0 + \frac{1}{2}\sum_k \boldsymbol{\Phi}_{jk}^1 q_{e,k}\right] q_{e,j}$
In both formulations, the equations of motion given in Appendix A lead to shape-integral expansions of the following form:

$$\boldsymbol{T} = \boldsymbol{T}^0 + \sum_{j=1..n_e} \boldsymbol{T}_j^1 q_{e,j} \tag{C2}$$

where $\boldsymbol{T}$ is a dummy variable standing for $\boldsymbol{M}_{\theta,\theta}$, $\boldsymbol{C}_t$, $\boldsymbol{C}_r$, $\boldsymbol{G}_r$, $\boldsymbol{G}_e$, or $\boldsymbol{O}_e$. The "0" and "1" terms are stored using a "Taylor" object-oriented class in the SID format defined by Wallrapp. The subtlety lays in the fact that the "1" terms will be different if the displacement is developed using a first order expansion or a second order expansion. Some terms involving $\boldsymbol{\Phi}_{jk}^1$ will be

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present in the latter case. The reader is referred to Wallrapp (1993) for a full description of the Taylor-expanded terms. Setting $\boldsymbol{\Phi}^1_{jk} = 0$ in these expressions will lead to the 1st-order shape integral approach of Shabana.

Appendix D: Equations of motion of simple wind turbine models

D1 Three degrees of freedom model of an onshore or fixed-bottom wind turbine

In this section we provide some intermediate values to obtain the equations of motion given in subsection 4.4. The degrees of freedom are $\boldsymbol{q} = (q, \psi)$. The kinematics of the tower (at its origin) are zero:

$$\boldsymbol{v}_{O,T} = \boldsymbol{0}, \quad \boldsymbol{\omega}_T = \boldsymbol{0}, \quad \boldsymbol{a}_{O,T} = \boldsymbol{0} \tag{D1}$$

All Jacobians are zero except $\boldsymbol{J}_{e,1T} = 1$ The inertial force, torque and elastic force are:

$$\boldsymbol{f}^*_T = C_{tTx}\ddot{q}\hat{\boldsymbol{t}}_x + M_T g\hat{\boldsymbol{t}}_z, \quad \boldsymbol{\tau}^*_T = C_{rTy}\ddot{q}\hat{\boldsymbol{t}}_y, \quad \boldsymbol{E}^*_T = f_e + D_e\dot{q} + (K_e + K_q)q + M_e\ddot{q} \tag{D2}$$

The nacelle kinematics (at its center of mass) are:

$$\boldsymbol{v}_{G,N} = \dot{q}\hat{\boldsymbol{t}}_x + \nu_y z_{NG}\dot{q}\hat{\boldsymbol{n}}_x - \nu_y x_{NG}\dot{q}\hat{\boldsymbol{n}}_z, \quad \boldsymbol{\omega}_N = \nu_y\dot{q}\hat{\boldsymbol{t}}_y \tag{D3}$$

$$\boldsymbol{a}_{G,N} = \ddot{q}\hat{\boldsymbol{t}}_x + (-\nu_y^2 x_{NG}\dot{q}^2 + \nu_y z_{NG}\ddot{q})\hat{\boldsymbol{n}}_x + (-\nu_y^2 z_{NG}\dot{q}^2 - \nu_y x_{NG}\ddot{q})\hat{\boldsymbol{n}}_z \tag{D4}$$

The Jacobians with respect to q are:

$$\boldsymbol{J}_{v,1N} = \hat{\boldsymbol{t}}_x + \nu_y z_{NG}\hat{\boldsymbol{n}}_x - \nu_y x_{NG}\hat{\boldsymbol{n}}_z, \quad \boldsymbol{J}_{\omega,1N} = \nu_y\hat{\boldsymbol{t}}_y \tag{D5}$$

The inertial force and torque on the nacelle are:

$$\boldsymbol{f}^*_N = M_N\ddot{q}\hat{\boldsymbol{t}}_x + M_N\left(-\nu_y^2 x_{NG}\dot{q}^2 + \nu_y z_{NG}\ddot{q}\right)\hat{\boldsymbol{n}}_x + M_N\left(-\nu_y^2 z_{NG}\dot{q}^2 - \nu_y x_{NG}\ddot{q}\right)\hat{\boldsymbol{n}}_z, \quad \boldsymbol{\tau}^*_N = J_{y,N}\nu_y\ddot{q}\hat{\boldsymbol{n}}_y \tag{D6}$$

The kinematics of the rotor are:

$$\boldsymbol{v}_{G,R} = \dot{q}\hat{\boldsymbol{t}}_x + \nu_y z_{NR}\dot{q}\hat{\boldsymbol{n}}_x - \nu_y x_{NR}\dot{q}\hat{\boldsymbol{n}}_z, \quad \boldsymbol{\omega}_R = \dot{\psi}\hat{\mathbf{e}}_{\mathbf{rx}} + \nu_y\dot{q}\hat{\boldsymbol{t}}_y \tag{D7}$$

$$\boldsymbol{a}_{G,R} = \ddot{q}\hat{\boldsymbol{t}}_x + (-\nu_y^2 x_{NR}\dot{q}^2 + \nu_y z_{NR}\ddot{q})\hat{\boldsymbol{n}}_x + (-\nu_y^2 z_{NR}\dot{q}^2 - \nu_y x_{NR}\ddot{q})\hat{\boldsymbol{n}}_z \tag{D8}$$

The corresponding Jacobians with respect to q ("1") and ψ ("2") are:

$$\boldsymbol{J}_{v,1R} = \hat{\boldsymbol{t}}_x + \nu_y z_{NR}\hat{\boldsymbol{n}}_x - \nu_y x_{NR}\hat{\boldsymbol{n}}_z, \quad \boldsymbol{J}_{\omega,1R} = \nu_y\hat{\boldsymbol{t}}_y, \quad \boldsymbol{J}_{\omega,2R} = \hat{\boldsymbol{r}}_x$$

The inertial force and torque on the rotor are:

$$\boldsymbol{f}^*_R = M_R\ddot{q}\hat{\boldsymbol{t}}_x + M_R\left(-\nu_y^2 x_{NR}\dot{q}^2 + \nu_y z_{NR}\ddot{q}\right)\hat{\boldsymbol{n}}_x + M_R\left(-\nu_y^2 z_{NR}\dot{q}^2 - \nu_y x_{NR}\ddot{q}\right)\hat{\boldsymbol{n}}_z \tag{D9}$$

$$\boldsymbol{\tau}^*_R = J_{x,R}\ddot{\psi}\hat{\boldsymbol{r}}_x \tag{D10}$$

$$+ (J_{\oplus,R}\nu_y\sin(\psi)\dot{\psi}\dot{q} + J_{\oplus,R}\left(-\nu_y\sin(\psi)\dot{\psi}\dot{q} + \nu_y\cos(\psi)\ddot{q}\right) - J_{x,R}\nu_y\sin(\psi)\dot{\psi}\dot{q})\hat{\boldsymbol{r}}_y \tag{D11}$$

$$+ (J_{\oplus,R}\nu_y\cos(\psi)\dot{\psi}\dot{q} + J_{\oplus,R}\left(-\nu_y\sin(\psi)\ddot{q} - \nu_y\cos(\psi)\dot{\psi}\dot{q}\right) - J_{x,R}\nu_y\cos(\psi)\dot{\psi}\dot{q})\hat{\boldsymbol{r}}_z \tag{D12}$$

D2 Three degrees of freedom model of an onshore or fixed-bottom wind turbine

The equations of motion for the model presented in subsection 4.5, with $\boldsymbol{q} = (q_1, q_2, \psi)$, are given in this section. The elements of the mass matrix are:

$$M_{11} = [M_{e11} + M_N + M_R] \tag{D13}$$
$$+ \left[J_{y,N} + J_{\oplus,R} + M_N\left(x_{NG}^2 - 2x_{NG}q_1 + z_{NG}^2\right) + M_R\left(x_{NR}^2 - 2x_{NR}q_1 + z_{NR}^2\right)\right]\nu_y^2 \tag{D14}$$
$$+ 2\left[M_N z_{NG} + M_R z_{NR}\right]\nu_y \tag{D15}$$
$$M_{13} = J_{x,R}\theta_t\nu_x\nu_y q_2 \tag{D16}$$
$$M_{22} = [M_{e22} + M_N + M_R] \tag{D17}$$
$$+ \left[J_{x,N} + J_{x,R} + M_N z_{NG}^2 + M_R z_{NR}^2\right]\nu_x^2 \tag{D18}$$
$$- 2\left[M_N z_{NG} + M_R z_{NR}\right]\nu_x \tag{D19}$$
$$M_{23} = J_{x,R}\nu_x \tag{D20}$$
$$M_{33} = J_{x,R} \tag{D21}$$

The elements of the forcing vector are:

$$f_1 = f_{e1} - K_{e11}q_1 - D_{e11}\dot{q}_1 - J_{x,R}\theta_t\nu_x\nu_y\dot{\psi}\dot{q}_2 + [M_N x_{NG} + M_R x_{NR}]\nu_y^2\dot{q}_1^2 \tag{D22}$$
$$+ g\left[M_N\left(\nu_y^2 z_{NG}q_1 + \nu_y x_{NG}\right) + M_R\left(\nu_y^2 z_{NR}q_1 + \nu_y x_{NR}\right)\right] + f_a\left[\theta_t\nu_y x_{NR} - \theta_t\nu_y q_1 + \nu_y z_{NR} + 1\right] \tag{D23}$$
$$f_2 = f_{e2} - K_{e22}q_2 - D_{e22}\dot{q}_2 + J_{x,R}\theta_t\nu_x\nu_y\dot{\psi}\dot{q}_1 \tag{D24}$$
$$+ g\left[M_N z_{NG} + M_R z_{NR}\right]\nu_x^2 q_2 + f_a\theta_t\nu_x q_2 \tag{D25}$$
$$f_3 = -J_{x,R}\theta_t\nu_x\nu_y\dot{q}_1\dot{q}_2 + \tau_a \tag{D26}$$

D3 Three degrees of freedom model of a floating wind turbine

The equations of motion for the model presented in subsection 4.6, with $\boldsymbol{q} = (x, \phi, q_T)$, are given in this section. The elements of the mass matrix are:

$$M_{11} = M_F + M_T + M_N \tag{D27}$$
$$M_{12} = M_F z_{FG} - M_{dTz} + M_N\left[L_T + z_{NG} - \nu_y x_{NG}q_T - \phi_y(x_{NG} + q_T + \nu_y z_{NG}q_T)\right] \tag{D28}$$
$$M_{13} = C_{tT1x} + M_N\left[1 + \nu_y z_{NG} - \nu_y^2 x_{NG}q_T - \phi_y(\nu_y^2 z_{NG}q_T + \nu_y x_{NG})\right] \tag{D29}$$
$$M_{22} = J_{y,F} + M_F z_{FG}^2 + J_{T,y} + J_{y,N} + M_N\left[(L_T^2 + z_{NG})^2 + (q_T + x_{NG})^2 + 2\nu_y q_T(z_{NG}q_T - L_T x_{NG})\right] \tag{D30}$$
$$M_{23} = C_{rT1y} + \left[J_{y,N} + M_N(x_{NG}^2 + z_{NG}^2 + L_T z_{NG} + \nu_y q_T(z_{NG}q_T - L_T x_{NG})\right]\nu_y + M_N\left[L_T + z_{NG}\right] \tag{D31}$$
$$M_{33} = M_e + M_N + \left[J_{y,N} + M_N\left(x_{NG}^2 - 2x_{NG}q_T + z_{NG}^2\right)\right]\nu_y^2 + 2M_N\nu_y z_{NG} \tag{D32}$$

The elements of the forcing vector are:

$$f_1 = f_H + [M_F z_{FG} - M_{dz} + M_N(L_T + z_{NG} - \nu_y x_{NG} q_T)]\,\phi_y \dot{\phi}_y^2 + M_N\,[q_T + x_{NG} + \nu_y z_{NG} q_T]\,\dot{\phi}_y^2 \tag{D33}$$

$$+ \left[2C_{tx} + M_N(1 + \nu_y z_{NG} - \nu_y^2 x_{NG} q_T)\right]\phi_y \dot{\phi}_y \dot{q}_T + M_N \nu_y\,[x_{NG} + \nu_y z_{NG} q_T]\,\dot{\phi}_y \dot{q}_T \tag{D34}$$

$$+ M_N \nu_y^2\,[x_{NG} + z_{NG}\phi_y]\,\dot{q}_T^2 \tag{D35}$$

$$+ f_a\,[1 - \theta_t \nu_y q_T - \nu_y \phi_y q_T] \tag{D36}$$

$$f_2 = \tau_H + M_N\left[\nu_y^2(L_T x_{NG} - z_{NG} q_T)\right]\dot{q}_T^2 \tag{D37}$$

$$- 2M_N\left[q_T + x_{NG} + \nu_y(2z_{NG} q_T - L_T x_{NG}) - \nu_y^2 q_T(L_T z_{NG} + x_{NG} q_T)\right]\dot{\phi}_y \dot{q}_T \tag{D38}$$

$$+ g\,[M_F z_{FG}\phi_y - M_{dz}\phi_y + M_N\{(L_T + z_{NG} - \nu_y x_{NG} q_T)\phi_y + q_T + x_{NG} + \nu_y z_{NG} q_T\}] \tag{D39}$$

$$+ f_a\left[L_T + z_{NR} + \theta_t x_{NR} + \theta_t q_T + \nu_y q_T^2 - L_T \theta_t \nu_y q_T\right] \tag{D40}$$

$$f_3 = f_e - D_e \dot{q}_T - K_e q_T \tag{D41}$$

$$+ M_N\left[q_T + x_{NG} + \nu_y(2z_{NG} q_T - L_T x_{NG}) - \nu_y^2 q_T(L_T z_{NG} + x_{NG} q_T)\right]\dot{\phi}_y^2 \tag{D42}$$

$$+ M_N \nu_y^2 x_{NG} \dot{q}_T^2 \tag{D43}$$

$$+ g\left[C_{tT1x}\phi_y + M_N\left(\nu_y x_{NG} + \nu_y^2 z_{NG} q_T - \nu_y^2 x_{NG}\phi_y q_T + \nu_y z_{NG}\phi_y + \phi_y\right)\right] \tag{D44}$$

$$+ f_a\,[1 + \theta_t \nu_y x_{NR} - \theta_t \nu_y q_T + \nu_y z_{NR}] \tag{D45}$$

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