March 6, 2025

The authors wish to thank the referee for the time and effort spent in reviewing the manuscript. The comments have helped to improve the paper and clarify important points that we discuss. Ours responses to the comments are included below.

Reviewer 2

The paper aims to develop a framework for modeling active wake farm mixing, with particular attention to the impacts of large scale coherent structures and turbulence on the mean flow. The model is interesting and provide a new way to analyze a promising approach for reducing wake effects. However, the literature review is incomplete and focuses on models not designed to capture the features of interest and other attempts to investigate coherent structures in the wakes, which actually makes the paper claims less compelling. Detailed comments regarding this point as well as other minor comments/questions follow.

1. The introduction and comparisons focus on the improvement of the model with respect to static wake models, which are not designed to capture dynamic behavior. There are a number of dynamic models that would serve as a better focus of both the literature review and comparisons.

   A paragraph has been added to the introduction to discuss dynamic wake models alongside the steady-state models that are commonly used for wind farm optimization. The primary focus is on data-driven representations of coherent flow structures, including models based on Proper Orthogonal Decomposition (POD), Resolvent Analysis, and Dynamic Mode Decomposition (DMD). The Dynamic Wake Meandering model is also briefly mentioned as another reduced-order modeling approach aimed at capturing unsteady dynamics.

2. Resolvant analysis has been recently applied to study wind farm wakes and it would be useful to compare this approach (or at least include it the literature review), i.e. on the top of page 3 where the authors mention that large-scale coherent structures have not been studied in this context. There have also been POD and DMD based studies of wind farm wakes that precisely aim to characterize coherent structures in wind farm wakes. DMD is in fact a dynamic approach.

   As mentioned in our response to the previous comment, we have added a paragraph on dynamic wake modeling to the introduction, focusing on data-driven representations of coherent flow structures. Three recent developments using Spectral SPOD, Resolvant Analysis, and DMD are discussed in the context of Active Wake Mixing. Additionally, we discuss the limitations of data-driven approaches and our rationale for pursuing an analytical representation of coherent flow structures instead.

3. The paper mentions the focus being on offshore and stable atmospheric conditions (line 70) but none of the results and model development are applicable to stable conditions.

This point should be clarified, in fact I suggest removing this statement since it does not accurately reflect the paper content (which clearly states the linear stability analysis does not include key effects of a stable boundary layer line 170)

We agree with the reviewer that the mention of stability conditions is not reflective of the current capabilities of the RANS and linear stability model. In the revised manuscript, we have clarified this statement so that it now refers to low TI conditions, which would be the likely environment where AWM could be applied.

4. Why is RANS the best approach for this work? Many RANS closure models are known to have some limitations simultaneously capturing both the mean flow and wave behavior and it would be useful to understand how/why the configuration selected overcomes these issues and why this approach is better than the alternatives.

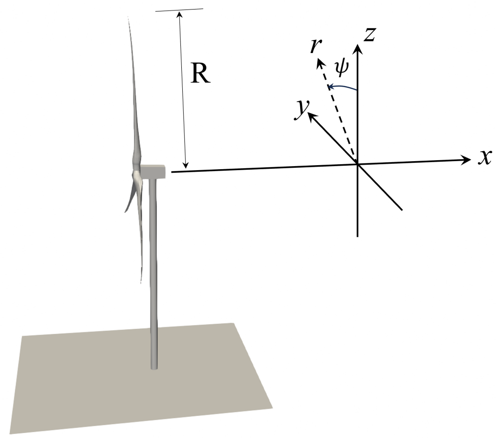
   We adapt a RANS approach since the primary goal for our model is to capture the effects of the coherent structures on the mean flow with a spatial linear stability formulation. Our results show that we can accomplish with the triple decomposition formulation as described in sections 2.2 and 2.3 where the large-scale coherent structures can impact the mean flow and vice-versa. We also want to note that our approach is neither recommending nor is limited to the $k-\epsilon$ turbulence model; rather, it represents one common approach to modeling the effects of turbulence. Our formulation is generalizable to most common RANS closure models.

5. There are a number of grammatical issues (another careful proof reading is likely to catch these)

   The revised manuscript has been reviewed for grammatical errors and typos and several corrections have been made.

6. The coordinate frame should be specified. The authors use y and r, clearly different coordinate frames are used, so clarification would be useful.

   In the revised manuscript, we have included a schematic in figure 2 (shown below) that shows both the cylindrical coordinate system used in this work, and its relationship to the Cartesian system relative to a typical turbine.



7. In many shear flows, singular values (e.g. resolvant modes or POD modes) provide more accurate characterization of the behavior of coherent structures, why are eigenvalues the best approach here?

Both the singular value decomposition and the eigenvalue decomposition offer useful representations of data, and they are often related. For instance, the reviewer mentions POD modes in connection to singular values but, in fact, the standard (space-only) POD is given by the solution to the Fredholm eigenvalue problem:

$$\int_{\Omega} \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{\phi}(\boldsymbol{x}')d\boldsymbol{x}' = \lambda \boldsymbol{\phi}(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{x}')$ is the two-point spatial correlation tensor. The solution to (1) is a set of eigenvectors, $\boldsymbol{\psi}$, and eigenvalues, λ, that represent the coherent flow structures and the average TKE captured by the flow structures, respectively. Therefore, eigenvalues are naturally associated with the coherent structures in a flow. A connection to singular values arises when solving (1) discretely. The analytical eigenvalue problem is represented discretely by a system of the form $\mathbf{C}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}$, where $\mathbf{C} = \mathbf{U}\mathbf{U}^H$ and $\mathbf{U} \in \mathbb{R}^{N_x \times N_t}$. Often, the number of points in time (N_t) that are used to form $\mathbf{C}$ is much less than the number of spatial points (N_x), and so this system is solved efficiently by performing a low-rank SVD of the matrix $\mathbf{U} = \mathbf{L}\boldsymbol{\Sigma}\mathbf{R}^H$. Using this decomposition, $\mathbf{C}$ can be expressed as $\mathbf{C} = \mathbf{L}\boldsymbol{\Sigma}^2\mathbf{L}$. Thus, the eigenvalues, λ, are given by the square of the singular values and the eigenvectors, $\boldsymbol{\psi}$, are given by the left singular vectors.

The connection between eigenvalues and singular values extends beyond computational efficiency. As the reviewer points out, the optimal inputs and outputs to the resolvent operator, $\mathcal{R}$, in a Resolvent-analysis is defined through the SVD, $\mathcal{R} = \sum_j \sigma_j \mathbf{u}_j \otimes \mathbf{v}_j$. Here, $\mathbf{v}_j$ are the input modes and $\mathbf{u}_j$ are the output modes, and the "gain" between input and output pairs are quantified by the square of the singular value σ_j^2. A connection between resolvent analysis and (Spectral) POD is obtained by expressing the Fourier transform of the two-point correlation tensor in terms of resolvent modes. In this case, the POD eigenvalues are found to be the square of the resolvant singular values (see Towne, Schmidt, and Colonius (2018) for more details).

In the work here, we focus on eigenvalues because they directly inform us about the stability characteristics of flow structures. Specifically, the solution to the Rayleigh equation formulated in Section 2.4.2 is obtained by solving the corresponding characteristic equation assuming a solution of the form $\tilde{\phi}(x, r, \psi, t) = \hat{\phi}_n(r) e^{i\alpha x + in\psi - i\omega t}$. Here, α are the eigenvalues that are associated with the linear operator defined by the differential equation. The real part of the eigenvalue determines the wavelength of the large-scale coherent structures, while the imaginary component of the eignenvalue quantifies the spatial growth of the structures.