Brief communication: A note on the variance of wind speed and turbulence intensity

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Abstract. This note addresses the issue that several papers in the peer-reviewed literature of wind energy applications have used an incorrect equation that equals the variance of wind speed (σ_U^2) to the sum of the variances of the wind components. This incorrect equation is often used to calculate turbulence intensity (TI), which, as a consequence, is often incorrectly estimated too. While exact analytical equations do not exist, here two approximate analytical equations are derived for σ_U^2 and TI, both functions of the variances of the wind components. Both formulations are validated with samples from a prior field campaign and perform satisfactorily.

1 Introduction

The standard deviation of wind speed, which is the square root of the variance, is an important parameter in wind energy applications because it is part of the definition of turbulence intensity used in the International Electrotechnical Commission (IEC) standard (International Electrotechnical Commission, 2019), which wind turbines must comply with. Since wind turbines always face the wind, especially in the first experiments that were conducted in wind tunnels and in idealized simulations, the convention has always been to align the x-axis along the mean wind direction. This convention is also adopted in boundary-layer meteorology, micrometeorology, and air pollution science, due to the focus on turbulence (Kaimal and Finnigan, 1994). With this convention, the variance of wind speed is accurately approximated as the variance of the u-component of the wind, i.e., the component along x. By contrast, in mesoscale meteorology and, more broadly, in geophysical applications, such as meteorological field campaigns or simulations of weather events, the convention is to align the x-axis along the east-west direction (and the y-axis along the north-south). Since the wind direction changes during the diurnal and seasonal cycles, the wind-aligned system of coordinates traditionally used in the wind energy community has become unpractical when studying real wind farms. The geophysical system of coordinates, therefore, has been adopted for field measurements and simulations of wind farms. As a consequence, the variance of wind speed is no longer a simple function of the variances of the wind components. While an exact analytical equation is impossible to obtain, an incorrect expression is often found in the literature, namely, the sum of the variances of the wind components, and often treated, incorrectly, as an exact definition (see for example Eq. 6 in Joffre and Laurila (1988)). Since turbulence intensity (TI) is defined in the IEC standard as the "ratio of the wind speed standard deviation to the mean wind speed" (International Electrotechnical Commission, 2019), errors are introduced in the calculation of turbulence intensity too. This note addresses this issue by proposing an analytical approximation for the wind speed variance and one for turbulence intensity for the geophysical system of coordinates.

The IEC standard is possibly the only case in which a single value of turbulence intensity is adopted. In most fields, three turbulence intensities are typically used, one for each direction ($TI_x = \sigma_u/\bar{U}$, and similarly for TI_y and TI_z), where x, y, and z are either the east-west, north-south, and vertical directions (e.g., in mesoscale meteorology) or the along-wind, cross-wind, and vertical directions (e.g., in micrometeorology, wind turbine design, or wind turbine load studies). The IEC definition of TI is also troubling because it does not specify which temporal scales should be considered in its calculation. Strictly speaking, turbulence intensity should refer only to fluctuations of the wind in the micro-scale (i.e., time averages of the order of minutes), thus to the right of the spectral gap in the wind spectrum. By contrast, wind fluctuations associated with meso or synoptic scale features belong to the left of the spectral gap and should not be called turbulent. In such cases, the ratio of the wind speed standard deviation over the mean, calculated over longer time intervals (i.e., hours to days), can still be obtained, but it should not be called a "turbulence" intensity. The equations derived here may be applied to any scale, but the focus is on the micro-scale.

The issue of the relationship between the standard deviation of wind speed and those of the wind components is relevant because turbulent kinetic energy, which is an important variable that is predicted by weather prediction models and that is affected by wind farms, is a function of the standard deviations of the wind components, while turbulence intensity – as defined in the IEC standard – is a function of the standard deviation of wind speed. Converting between the two is therefore important in validating model results against observations.

2 Definitions

Let us use the geophysical system of coordinates. The wind components along x, y and z are u, v and w, respectively, and the magnitude U is a non-linear function of all three:

$$U = f(u,v,w) = \sqrt{u^2 + v^2 + w^2}. \tag{1}$$

The means $\bar{u}$, $\bar{v}$, $\bar{w}$,and $\bar{U}$, calculated over a set of N measurements u_t, v_t, w_t, and U_t, each taken at time t, are:

$$\bar{u} = \frac{1}{N}\sum_t u_t, \quad \bar{v} = \frac{1}{N}\sum_t v_t, \quad \bar{w} = \frac{1}{N}\sum_t w_t, \tag{2}$$

$$\bar{U} = \frac{1}{N}\sum_t U_t. \tag{3}$$

The variances σ_u^2, σ_v^2, σ_w^2, and σ_U^2 are:

$$\sigma_u^2 = \frac{1}{N}\sum_t (u_t - \bar{u})^2 = \overline{(u-\bar{u})^2} = \overline{u^2 - 2u\bar{u} + \bar{u}^2} = \overline{u^2} - 2\bar{u}^2 + \bar{u}^2 = \overline{u^2} - \bar{u}^2, \tag{4}$$

$$\sigma_v^2 = \frac{1}{N}\sum_t (v_t - \bar{v})^2 = \overline{v^2} - \bar{v}^2, \quad \sigma_w^2 = \frac{1}{N}\sum_t (w_t - \bar{w})^2 = \overline{w^2} - \bar{w}^2, \tag{5}$$

$$\sigma_U^2 = \frac{1}{N}\sum_t \left(U_t - \bar{U}\right)^2 = \overline{U^2} - \bar{U}^2 = \overline{u^2} + \overline{v^2} + \overline{w^2} - \bar{U}^2 = \sigma_u^2 + \sigma_v^2 + \sigma_w^2 + \bar{u}^2 + \bar{v}^2 + \bar{w}^2 - \bar{U}^2. \tag{6}$$

Eq. 6 may not be simplified analytically any further because:

$$\bar{U}^2 = \left(\overline{\sqrt{u^2 + v^2 + w^2}}\right)^2 \neq \bar{u}^2 + \bar{v}^2 + \bar{w}^2. \tag{7}$$

As a consequence:

$$\sigma_U^2 \neq \sigma_u^2 + \sigma_v^2 + \sigma_w^2 \tag{8}$$

and

$$TI^2 = \frac{\sigma_U^2}{\bar{U}^2} \neq \frac{\sigma_u^2 + \sigma_v^2 + \sigma_w^2}{\bar{U}^2} = \frac{2\,TKE}{\bar{U}^2}, \tag{9}$$

where:

$$TKE = \frac{\sigma_u^2 + \sigma_v^2 + \sigma_w^2}{2}. \tag{10}$$

In order to obtain an expression for the variance of wind speed, we first need to recognize that the wind is intrinsically turbulent and therefore we can use the Reynolds averaging approach. The turbulent fluctuations, usually denoted with a prime ($'$), in this case coincide exactly with the differences from the means (δ) as follows:

$$u_t = \bar{u} + u_t' = \bar{u} + \delta u_t, \tag{11}$$

and similarly for v_t, w_t and U_t. Therefore the variances can be rewritten exactly as:

$$\sigma_u^2 = \frac{1}{N}\sum u_t'^2 = \frac{1}{N}\sum (\delta u_t)^2 = \overline{u'^2} = \overline{(\delta u)^2}, \tag{12}$$

and similarly for σ_v^2, σ_w^2, and σ_U^2.

3 Proposed formulation

Following the approach of Ackermann (1983) and Baird (1962), we introduce the only approximation of this manuscript: that the δ's coincide with the differentials. This is equivalent to assuming that the fluctuations (and the δ's) are smaller in magnitude than their respective means, which is realistic, but may or may not be true in all atmospheric conditions. The goal is to derive formulations for σ_U^2 and TI that depend only on statistics of the wind components.

First, we use the assumption that the δ's can be approximated as differentials as follows:

$$(\delta U)^2 \approx \left(\frac{\partial U}{\partial u}\right)^2 (\delta u)^2 + \left(\frac{\partial U}{\partial v}\right)^2 (\delta v)^2 + \left(\frac{\partial U}{\partial w}\right)^2 (\delta w)^2 \tag{13}$$

$$+2\left(\frac{\partial U}{\partial u}\right)\left(\frac{\partial U}{\partial v}\right)\delta u \delta v + 2\left(\frac{\partial U}{\partial u}\right)\left(\frac{\partial U}{\partial w}\right)\delta u \delta w + 2\left(\frac{\partial U}{\partial v}\right)\left(\frac{\partial U}{\partial w}\right)\delta v \delta w.$$

Note that, in Eq. 13, the partial derivatives are to be evaluated at the "point" of the function $U = f(u, vw)$ around which there are the fluctuations, thus for the mean values $\bar{u}$,$\bar{v}$, and $\bar{w}$. The three partial derivatives are therefore:

$$\left(\frac{\partial U}{\partial u}\right) = \left.\frac{\partial U}{\partial u}\right|_{\bar{u},\bar{v},\bar{w}} = \left.\frac{1}{2}\left(u^2+v^2+w^2\right)^{-\frac{1}{2}}(2u)\right|_{\bar{u},\bar{v},\bar{w}} = \frac{\bar{u}}{\sqrt{\bar{u}^2+\bar{v}^2+\bar{w}^2}}, \tag{14}$$

$$\left(\frac{\partial U}{\partial v}\right) = \left.\frac{\partial U}{\partial v}\right|_{\bar{u},\bar{v},\bar{w}} = \left.\frac{1}{2}\left(u^2+v^2+w^2\right)^{-\frac{1}{2}}(2v)\right|_{\bar{u},\bar{v},\bar{w}} = \frac{\bar{v}}{\sqrt{\bar{u}^2+\bar{v}^2+\bar{w}^2}}, \tag{15}$$

$$\left(\frac{\partial U}{\partial w}\right) = \left.\frac{\partial U}{\partial w}\right|_{\bar{u},\bar{v},\bar{w}} = \left.\frac{1}{2}\left(u^2+v^2+w^2\right)^{-\frac{1}{2}}(2w)\right|_{\bar{u},\bar{v},\bar{w}} = \frac{\bar{w}}{\sqrt{\bar{u}^2+\bar{v}^2+\bar{w}^2}}, \tag{16}$$

which are not a function of time t. Replacing Eqs. 14–16 into Eq. 13 leads to the following expression for σ_U^2:

$$\sigma_U^2 = \frac{1}{N}\sum(\delta U)^2 = \overline{(\delta U)^2} \approx \frac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2 + \bar{w}^2\sigma_w^2 + 2\bar{u}\bar{v}\sigma_{uv} + 2\bar{u}\bar{w}\sigma_{uw} + 2\bar{v}\bar{w}\sigma_{vw}}{\bar{u}^2+\bar{v}^2+\bar{w}^2}, \tag{17}$$

where σ_{uv}, σ_{uw}, and σ_{vw} are the covariances of u and v, u and w, and v and w, respectively, which can be positive or negative.

To obtain an expression for TI, we derive an approximation for $\overline{U}$ as follows:

$$\bar{U} = \overline{\sqrt{(\bar{u}+u')^2+(\bar{v}+v')^2+(\bar{w}+w')^2}} \tag{18}$$

$$= \sqrt{\bar{u}^2+\bar{v}^2+\bar{w}^2}\,\overline{\sqrt{\frac{(\bar{u}+u')^2+(\bar{v}+v')^2+(\bar{w}+w')^2}{\bar{u}^2+\bar{v}^2+\bar{w}^2}}} \tag{19}$$

$$= \sqrt{\bar{u}^2+\bar{v}^2+\bar{w}^2}\,\overline{\sqrt{1+\frac{2u'\bar{u}+u'^2+2v'\bar{v}+v'^2+2w'\bar{w}+w'^2}{\bar{u}^2+\bar{v}^2+\bar{w}^2}}}. \tag{20}$$

The term under the square root can be simplified via the binomial approximation for $\alpha = 1/2$:

$$(1+x)^\alpha \approx (1+\alpha x), \tag{21}$$

valid for $|x| < 1$ and $|\alpha x| << 1$, which are generally true in Eq. 20 due to the assumption that the fluctuations are small with respect to the means, as follows:

$$\bar{U} \approx \sqrt{\bar{u}^2+\bar{v}^2+\bar{w}^2}\,\overline{1+\frac{u'\bar{u}+v'\bar{v}+w'\bar{w}}{\bar{u}^2+\bar{v}^2+\bar{w}^2}+\frac{1}{2}\frac{u'^2+v'^2+w'^2}{\bar{u}^2+\bar{v}^2+\bar{w}^2}}. \tag{22}$$

Using the Reynolds averaging properties, the final expressions for $\overline{U}$ and $\overline{U}^2$ are:

$$\bar{U} \approx \sqrt{\bar{u}^2+\bar{v}^2+\bar{w}^2}\left(1+\frac{1}{2}\frac{\sigma_u^2+\sigma_v^2+\sigma_w^2}{\bar{u}^2+\bar{v}^2+\bar{w}^2}\right), \tag{23}$$

$$\bar{U}^2 \approx \left(\bar{u}^2 + \bar{v}^2 + \bar{w}^2\right)\left(1 + \frac{1}{2}\frac{\sigma_u^2 + \sigma_v^2 + \sigma_w^2}{\bar{u}^2 + \bar{v}^2 + \bar{w}^2}\right)^2. \tag{24}$$

Since the term in parenthesis in Eq. 24 is greater than 1, not only is the inequality in Eq. 7 confirmed, but it can be further expanded to:

$$\bar{U}^2 > \bar{u}^2 + \bar{v}^2 + \bar{w}^2. \tag{25}$$

One could be tempted to replace the expression for $\bar{U}^2$ from Eq. 24 in Eq. 6, but doing so would cause the expression for the variance of wind speed to become negative because the error introduced by the binomial approximation, although small when used for $\bar{U}$, is amplified in $\bar{U}^2$, especially when it is used in a difference of terms of similar magnitudes as in Eq. 6. When used in the denominator and alone, however, as is the case for TI from Eq. 9, Eq. 24 is acceptable and we obtain:

$$TI^2 = \frac{\sigma_U^2}{\overline{U}^2} \approx \frac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2 + \bar{w}^2\sigma_w^2 + 2\bar{u}\bar{v}\sigma_{uv} + 2\bar{u}\bar{w}\sigma_{uw} + 2\bar{v}\bar{w}\sigma_{vw}}{\left(\bar{u}^2 + \bar{v}^2 + \bar{w}^2\right)^2\left(1 + \frac{1}{2}\frac{\sigma_u^2 + \sigma_v^2 + \sigma_w^2}{\bar{u}^2 + \bar{v}^2 + \bar{w}^2}\right)^2}. \tag{26}$$

To simplify the notation without loosing generality, we hereafter assume that the wind is a two-dimensional vector. This assumption is often used in mesoscale meteorology and is needed when only 2D measurements of the wind are available (e.g., with a cup anemometer). Thus all terms that are a function of w drop from Eq. 17:

$$\sigma_U^2 \approx \frac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2 + 2\bar{u}\bar{v}\sigma_{uv}}{\bar{u}^2 + \bar{v}^2} < \sigma_u^2 + \sigma_v^2. \tag{27}$$

Using $\sigma_u^2 + \sigma_v^2$ as an approximation for σ_U^2 generally causes an over-estimation of the variance of U, especially when $\bar{u}$ and $\bar{v}$ are of opposite sign (e.g., in the second and fourth quadrants) and the covariance is positive, or vice versa when $\bar{u}$ and $\bar{v}$ are of the same sign and σ_{uv} is negative.

If the two variables u, v were independent (but they are not), their covariance σ_{uv} would be zero; since σ_{uv} is often unknown, it can be set to zero as an approximation, to give an expression that is still overestimated by the sum of the wind component variances:

$$\sigma_U^2 \approx \frac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2}{\bar{u}^2 + \bar{v}^2} < \sigma_u^2 + \sigma_v^2. \tag{28}$$

Note that, when the x-direction is aligned along the mean wind, $\bar{v} = 0$ and therefore $\sigma_U^2 \approx \sigma_u^2$ from Eq. 27, consistent with the IEC convention (in which it is called σ_1^2) and further supported in the derivation in Appendix B by Larsén (2022).

Similarly for TI with 2D wind vectors, Eq. 26 becomes:

$$TI^2 \approx \frac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2 + 2\bar{u}\bar{v}\sigma_{uv}}{\left(\bar{u}^2 + \bar{v}^2\right)^2\left(1 + \frac{1}{2}\frac{\sigma_u^2 + \sigma_v^2}{\bar{u}^2 + \bar{v}^2}\right)^2} < \frac{\sigma_u^2 + \sigma_v^2}{\bar{u}^2 + \bar{v}^2}. \tag{29}$$

If the approximation for σ_U^2 from Eq. 28 and that for $\bar{U}$ from Eq. 7 are used, then:

$$TI^2 \approx \frac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2}{\left(\bar{u}^2 + \bar{v}^2\right)^2} < \frac{\sigma_u^2 + \sigma_v^2}{\bar{u}^2 + \bar{v}^2}. \tag{30}$$

4 Application

Wind measurements collected with a 20-Hz sonic anemometer mounted at 4 m during the American WAKE experimeNt (AWAKEN) field campaign (Atmosphere to Electrons (A2e), 2025), conducted in northern Oklahoma (U.S.A.) around five wind farms between 2022 and 2024, are used to demonstrate the validity of the proposed formulations and compare their performance against that of the inexact equations discussed above. A one-week period (23–29 July 2023) is selected for the analysis (Figure 1d).

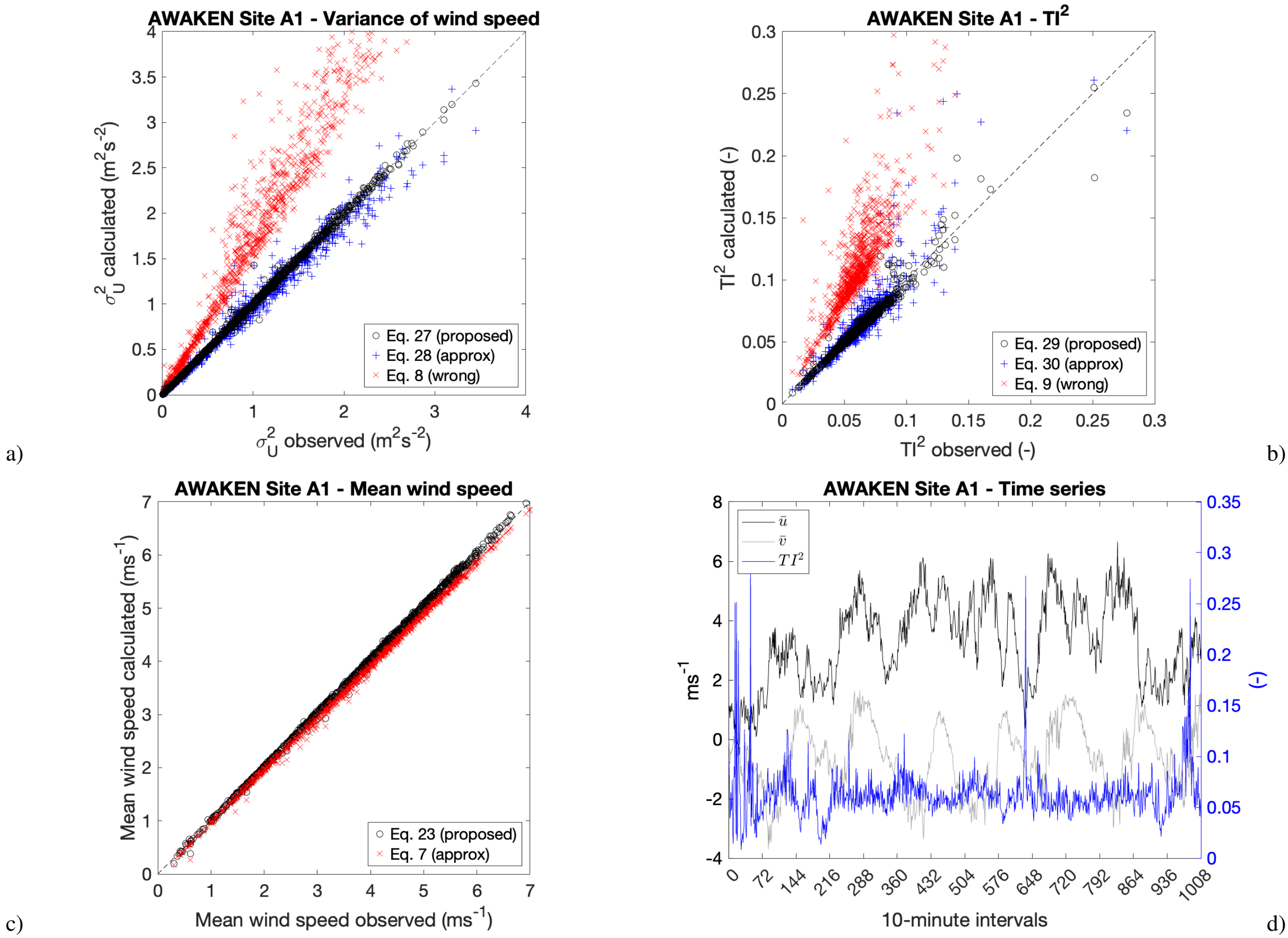

Figure 1. Scatter plots of 10-minute statistics from the AWAKEN campaign during the week of 23–29 July 2023: a) wind speed variance; b) turbulence intensity (squared); and c) mean wind speed. The time series of observed mean wind components and turbulence intensity (squared) are in d).

The proposed formulations for σ_U^2 (Eq. 27), TI^2 (Eq. 29), and $\bar{U}$ (Eq. 23) perform very well, with a very close alignment with the 1:1 line (Figure 1, a–c). For the variance, the mean absolute percent error (MAPE) is 2.4% for the proposed formulation, while using $\sigma_u^2 + \sigma_v^2$ (Eq. 28) always causes an overestimation (i.e., positive error), with a MAPE of 78.6% and a large positive bias of 0.70 m^{2}s^{-2} (Table 1). The MAPE for TI^2 with Eq. 29 is 3.7%, slightly larger than that for σ_U^2, due to the additional approximation introduced by the division of Eq. 27 by Eq. 24. TI is always grossly overestimated by using the approximation from Eq. 9 (MAPE = 95.1%), because the numerator overestimates, while the denominator slightly underestimates.

Table 1. Error analysis of the various equations analyzed in the manuscript.

EQUATION		BIAS	RMSE	MAPE
Eq. 27 (proposed):	$\sigma_U^2 \approx \dfrac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2 + 2\bar{u}\bar{v}\sigma_{uv}}{\bar{u}^2 + \bar{v}^2}$	1.2×10^{-3} m^{2}s^{-2}	0.03 m^{2}s^{-2}	2.4%
Eq. 28 (approx):	$\sigma_U^2 \approx \dfrac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2}{\bar{u}^2 + \bar{v}^2}$	-0.03 m^{2}s^{-2}	0.10 m^{2}s^{-2}	7.9%
Eq. 8 (wrong):	$\sigma_U^2 \neq \sigma_u^2 + \sigma_v^2$	0.70 m^{2}s^{-2}	0.88 m^{2}s^{-2}	78.6%
Eq. 29 (proposed):	$TI^2 \approx \dfrac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2 + 2\bar{u}\bar{v}\sigma_{uv}}{(\bar{u}^2 + \bar{v}^2)^2\left(1 + \dfrac{1}{2}\dfrac{\sigma_u^2 + \sigma_v^2}{\bar{u}^2 + \bar{v}^2}\right)^2}$	-3.3×10^{-4}	0.01	3.7%
Eq. 30 (approx):	$TI^2 \approx \dfrac{\bar{u}^2\sigma_u^2 + \bar{v}^2\sigma_v^2}{(\bar{u}^2 + \bar{v}^2)^2}$	4.1×10^{-3}	0.04	10.5%
Eq. 9 (wrong):	$TI^2 \neq \dfrac{\sigma_u^2 + \sigma_v^2}{(\bar{u}^2 + \bar{v}^2)^2}$	0.06	0.18	95.1%
Eq. 23 (proposed):	$\bar{U} \approx \sqrt{\bar{u}^2 + \bar{v}^2}\left(1 + \dfrac{1}{2}\dfrac{\sigma_u^2 + \sigma_v^2}{\bar{u}^2 + \bar{v}^2}\right)$	0.03 ms^{-1}	0.05 ms^{-1}	1.2%
Eq. 7 (approx):	$\bar{U} \approx \sqrt{\bar{u}^2 + \bar{v}^2}$	-0.09 ms^{-1}	0.10 ms^{-1}	2.5%

5 Conclusions

An analytical equation that approximates the variance of wind speed as a function of the variances of the wind components (in geophysical coordinates) is derived under the only assumption that the turbulent fluctuations of the wind components are small with respect to their means. The approximation for the variance of wind speed is then used, after a few steps, to derive another approximation for turbulence intensity. Although a through validation is beyond the scope of this note, both formulations appear to perform well for a few samples of observations obtained during the AWAKEN field campaign of 2023 and to outperform the two incorrect equations that have been used at times in the literature.

Competing interests. Archer is a member of the editorial board of Wind Energy Science.

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