Reply to Reviewer #1's Comments

This paper addresses the geometrically nonlinear analysis of Timoshenko beam structures with variable cross-sections, which remains a challenging problem in computational structural mechanics. The authors propose a co-rotational finite element framework that incorporates an analytically derived displacement-based Timoshenko beam element for variable cross-sections, with the aim of improving both accuracy and computational efficiency in modeling large-deformation behavior.

The work is generally well developed and addresses a relevant research topic, with references that are consistent with the adopted methodology. Nonetheless, some concerns remain and should be adequately addressed.

- Although the modeling of Timoshenko co-rotational beams with tapered or variable cross-sections has been addressed in previous studies using multiple successful approaches, it remains unclear what constitutes the specific novelty or methodological advancement in the present work. Could the authors clarify how their formulation provides a substantive improvement or development over existing models?

Reply:

Thank you for your comment. Our improvements to the large-deformation calculation method for variable-cross-section beams are primarily reflected in the following three aspects:

Improvement 1:

We propose a variable-cross-section Timoshenko beam element based on analytical displacement shape functions, which replaces traditional interpolation shape functions. This significantly enhances the computational accuracy of the element in geometrically nonlinear analysis.

Improvement 2:

Within the corotational framework, Gaussian integration is introduced to compute the stiffness and mass matrices of variable-cross-section beams. This avoids the repeated calculation of the moment of inertia for each cross-section required in conventional approaches, thereby improving computational efficiency. The method provides accurate and efficient solutions for beams with linearly varying width or thickness. For problems with pronounced taper—such as nonlinear variations in both width and thickness—additional section information is required to determine the additional unknown coefficients.

Improvement 3:

We propose a coordinate transformation method between local and global systems tailored for variable-cross-section beams. This approach can handle irregular sections and proportionally graded sections, extending the applicability of the corotational formulation.

- What is the practical advantage of incorporating analytically derived displacement shape functions within the co-rotational Timoshenko beam formulation, and how does this choice improve accuracy, computational efficiency, or overall performance compared to existing approaches?

Reply:

We thank the reviewer for the insightful comment. Compared with conventional interpolation-based shape functions, the analytically derived displacement shape functions—rigorously obtained from the equilibrium equations of the Timoshenko beam—offer the following three practical advantages:

(1)Improved Accuracy: Conventional interpolated shape functions introduce truncation errors, whereas the analytical shape functions can more accurately capture the actual bending deformation of the beam, thereby significantly enhancing computational accuracy in geometrically nonlinear analysis.

(2)Higher Computational Efficiency: The analytical shape functions can be directly incorporated into Gaussian quadrature, which reduces the required number of integration points and increases the efficiency in computing the element stiffness and mass matrices.

(3)Enhanced Numerical Stability: When dealing with problems involving large deformations and large rotations, the analytical shape functions better preserve the internal force equilibrium within the element, leading to improved overall numerical stability.
To further validate these points, we have added a new numerical example to demonstrate the reasonableness of the approach. The specific example is provided in the following reply.

- While the methodology presented in Section 2 attempts to account for geometric variability through Gaussian integration (Equation 16), there is a fundamental concern regarding the mathematical consistency of the local stiffness matrix derivation. The formulation in Equations(14 and 15) utilizes analytical shape functions originally developed for prismatic members; however, applying these functions to non-prismatic elements without incorporating the spatial derivatives of the sectional properties (EI'(x) and GA'(x)) introduces a known field inconsistency. For elements with significant tapering, the omission of these gradient terms may lead to an inaccurate representation of the internal equilibrium, potentially affecting the overall robustness of this co-rotational framework.

Reply:
Thank you for the reviewer insightful comments. Following your suggestion, we have added explanatory text in the corresponding section of the paper. The details are provided below.
In the reference by Friedman and Kosmatka (1993), the variation of the moment of inertia and area of a non-uniform beam is linearly interpolated using a taper coefficient. Building on this, Nguyen (2013) further employs higher-order interpolation of area and moment of inertia based on the taper coefficient to calculate the deformation of tapered beams. In this paper, we reduce the number of unknowns by assuming linear variations in width and thickness. The corresponding unknown coefficients are determined using **Eq. 19** and then substituted into **Eq. 18** to compute the area and moment of inertia of the varying cross-sections. This approach provides relatively accurate results for beams with simple taper.
When addressing large deformations of beams with significant taper, however, higher-order interpolation of width and thickness becomes necessary. Each additional interpolation order introduces two more unknowns, which means that additional known conditions—such as other cross-sectional dimensions or explicit expressions for thickness and diameter along the beam length—are required. Only with such information can the shape function expressions accurately describe the large deformation behavior of beams with pronounced taper.

- While the tangent stiffness formulation in Section 3 explicitly identifies the Km and Kg matrices (as seen in Equation 42), the derivation lacks a detailed discussion on how these components are specifically adapted to the element's variable cross-section.

Reply:
We appreciate the reviewer's comment. The global stiffness matrix Kg in the overall coordinate system is composed of two parts: the linear stiffness Ka and the nonlinear stiffness Km. The linear stiffness Ka is formed from the element stiffness matrix Ke in the local coordinate system, and Ke inherently contains cross-section variation information. The nonlinear stiffness term is related to the internal forces at the current iteration step and the transformation matrix. The transformation matrix depends only on the degrees of freedom at the element's end nodes, which has the same effect for both variable and constant cross-sections. The internal force vector is related to the element stiffness matrix Ke in the local coordinate system and the nodal displacements, and this Ke also inherently contains cross-section variation information.

- Although the proposed formulation is applied to six numerical examples, the study does not provide a comparative assessment against existing methods for variable cross-section beams, which limits the demonstration of the approach's relative effectiveness and advantages.

Reply:

Thank you for your comment. As per your suggestion, the corresponding section has been revised. This paper primarily compares the proposed calculation method for large deformation of variable-cross-section beams through three typical numerical examples (Examples 3–5), as detailed below:

Example 3 examines a uniformly tapered variable-cross-section beam. The traditional method used in the references divides the beam into segments of uniform cross-sections for analysis. Compared to this approach, when the same number of elements is used, the results obtained by the method proposed in this paper are more accurate.

Example 4 refers to the large deformation solution for variable-cross-section beams proposed by Nguyen et al., which combines the corotational formulation. In their approach, the cross-sectional area and moment of inertia are treated as linear functions of the taper coefficient. The results from this example show that the method proposed in this paper achieves higher computational accuracy compared to Nguyen's method.

Example 5 involves the large deformation analysis of a non-uniformly tapered variable-cross-section frame. In the reference (Araujo et al., 2017), at the element level, a flexibility system of equations based on the principle of virtual forces (PVF) is established to calculate the tangent stiffness matrix and the equivalent nodal loads. The example results indicate that the method proposed in this paper achieves comparable computational accuracy to that of the referenced method.

- In Section 4, while the first two examples address 3D configurations, they are limited to constant cross-sections, whereas the subsequent four examples that account for variable sections are restricted to 2D analyses. The absence of a 3D example with a variable cross-section represents a significant gap, as it leaves the element's performance unverified in cases where spatial geometric nonlinearity and sectional tapering are coupled. Consequently, the authors must explicitly define the intended scope of this formulation and specify the categories of structural problems it is reliably applicable to, as the current results do not yet justify its robustness for complex, non-prismatic 3D applications.

Reply:

We thank the reviewer for the constructive comment. To verify the correctness of the proposed method for spatial deformation analysis of variable-cross-section beams and to demonstrate the robustness of the algorithm, the numerical example presented in the reference by Murín and Kutiš (2002) has been computed and validated. The corresponding results have been added to the **Section 4.5** in **Applications** of the paper.

Figure 1 shows a 3D frame, with beams of varying circular cross-sections, loaded by concentrated loads F at node 1. The displacements of nodes 1 to 4 were founded. Variation of the cross-sectional area of the beams a is defined by the following diameter quadratic function $d(y) = 0.04 + 0.04y^2$. The beams b and c have constant diameters through lengths of elements. Detailed parameters can be found in (Murín et al. 2002). Only one exact beam element was used to model each beam (a, b, c). In the Hermite beam element model, only one element was used to represent the beams b and c in all cases, but beams a were modelled with 1, 2 and 3 elements in models 1, 2 and 3 respectively.

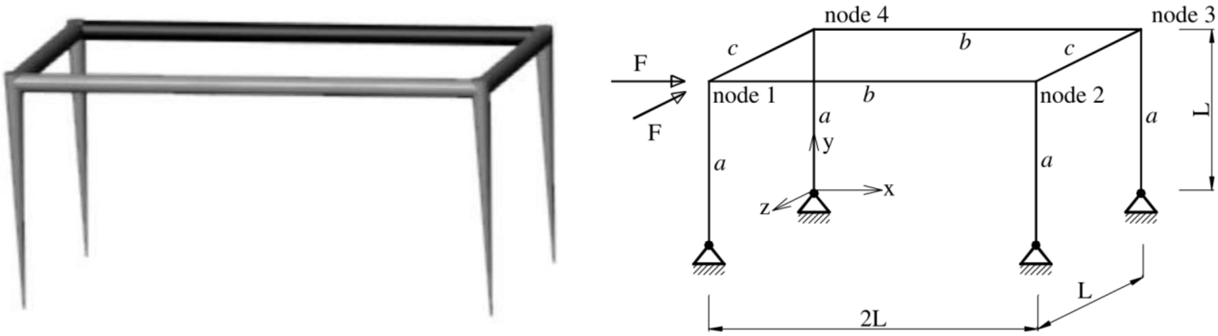

Figure 1. Frame displacement at node 13 **(Murín et al. 2002)**

The numerical results obtained by the present method are compared against those from the method proposed by Murín et al. (2002) and the solutions from classical Hermite beam elements, as presented in **Table 1**. It can be observed from the table that compared to the reference method, the displacement solutions of the present method at all nodes and under all loading cases are consistently closer to the exact solution, demonstrating a significant enhancement in computational accuracy. Furthermore, when the number of elements is varied, the present method exhibits a narrower and more stable variation range in its solutions, highlighting its superior numerical robustness.

Table 1. Comparison of results

	Node1 (errors %)		Node2 (errors %)		Node3 (errors %)		Node4 (errors %)	
	U_x(mm)	U_z(mm)	U_x(mm)	U_z(mm)	U_x(mm)	U_z(mm)	U_x(mm)	U_z(mm)
Exact solution	0.775	-1.098	0.774	-0.428	0.945	-0.428	0.945	-1.098
Model1 ref	0.651(16.0)	-0.882(19.7)	0.650	-0.336(21.5)	0.763(19.3)	-0.336	0.763	-0.882
Model1 this paper	0.745(3.9)	-0.981(10.7)	0.745	-0.427(0.2)	0.859(9.1)	-0.427	0.859	-0.981
Model2 ref	0.743(4.1)	-1.008(8.2)	0.722	-0.390(8.9)	0.869(8.0)	-0.390	0.869	-1.008
Model2 this paper	0.767(1.0)	-1.085(1.2)	0.766	-0.423(1.2)	0.933(1.3)	-0.423	0.933	-1.086
Model3 ref	0.749(3.4)	-1.054(4.0)	0.748	-0.409(4.4)	0.908(3.9)	-0.409	0.908	-1.054
Model3 this paper	0.772(0.4)	-1.093(0.5)	0.771	-0.426(0.5)	0.940(0.5)	-0.426	0.940	-1.093

Ref:

[1] Murín, Justín, and Vladimír Kutiš. "3D-beam element with continuous variation of the cross-sectional area." Computers & structures 80.3-4 (2002): 329-338.

- The abstract and introduction must explicitly define the research's novel contribution, clearly justifying the necessity of the proposed method instead of overemphasizing well-established classical theories

Reply:

Thank you for your comment. The abstract, introduction, and conclusion sections of the manuscript have been revised accordingly to better highlight the contributions of this study.