Reply to Reviewer #2's Comments

The authors formulate a co-rotational Timoshenko beam element for large displacement analysis of 3D frames, considering the influence of variable cross-section. The solution of the equilibrium equations is used to interpolate the lateral displacements and rotations for improving the element efficiency. Numerical examples and experiments are carried out to confirm the accuracy of the element. The topic is of interest, but many issues should be clarified for further evaluation of the paper.

- Primarily, the solution of the equilibrium equations of a Timoshenko beam with variable cross-section is difficult to derive. The author should provide more details on the derivation of the solution and the considered section profile. I believe that a solution for a general section cannot be derived. Additionally, the number of Gauss points used to evaluate the element tangent stiffness and mass matrices depends on the profile of the beam section. It is difficult to evaluate the element without knowing this information.

Reply:

Thank you for the suggestions regarding these two key issues. In fact, Friedman, Zack, and J. B. Kosmatka et al. (1993) studied the solution method for the equilibrium equations of tapered Timoshenko beams. They performed linear interpolation for the variations of the beam's moment of inertia and cross-sectional area using a taper coefficient. For details, please refer to the paper (Le et al., 2011) exceeding "Exact stiffness matrix of a nonuniform beam—II. Bending of a Timoshenko beam." Computers & structures 49.3 (1993): 545–555. Building on this, Nguyen et al. (2013) calculated the deformation of tapered beams using higher-order interpolation of area and moment of inertia.

In this paper, the calculation of the cross-sectional variation follows two specific approaches:

(1) When only the moments of inertia and cross-sectional areas at certain sections are known. This paper reduces the number of undetermined coefficients by assuming linear variations in width and thickness, which is relatively accurate for simple tapered beams. However, when calculating large deformations of beams with significantly tapered cross-sections, higher-order interpolation for width and thickness is required. Each additional order introduces two more undetermined coefficients, meaning additional known conditions (such as cross-sectional areas and moments of inertia in the y and z directions at other sections) are necessary. Only then can the shape function expressions accurately represent large deformations of beams with significant taper.

(2) When the specific expressions for cross-sectional dimension variations (such as width or diameter) are known. This paper directly calculates the cross-sectional characteristics at the corresponding Gaussian integration points of the element using these expressions, followed by solution via Gaussian integration. In this case, the method achieves high accuracy and good robustness even for nonlinearly varying cross-sectional dimensions. Following your suggestion, relevant validation examples have been added in **Section 4.5 (Applications)** of the paper.

Regarding the number of Gaussian integration points depending on the variation of the beam cross-section, we have conducted extensive research. For example, in the five examples in the paper, we tested with 2, 4, 6, and 8 integration points, covering cases such as uniform cross-sections, uniformly tapered cross-sections, complex tapered cross-sections, and spatial deformations of tapered beams. The results showed little difference. Therefore, all examples in this paper were computed using four Gaussian integration points.

- The authors claim that "the proposed method achieves both high computational efficiency and accuracy in handling large deformations and nonlinear behavior", but the efficiency is not demonstrated in the paper.

Reply:

Thank you for your comment. This is indeed a crucial issue. The computational process within each element in the proposed method differs only in the formulation of the relevant matrices, with no significant change in the computational efficiency of a single element. However, the proposed method can achieve relatively accurate results even with fewer element divisions, thereby improving the overall computational efficiency of the algorithm to some extent. To validate this conclusion, a numerical example from the reference Murın, J., Justın, V., & Kutiš, V. (2002). "3D-beam element with continuous variation of the cross-sectional area." Computers & structures, 80(3-4), 329-338, was calculated and added to the Applications section of the paper. The results show that using only one element can achieve computational accuracy comparable to that obtained with three elements in the referenced study. The corresponding results have been added to the **Section 4.5** in **Applications** of the paper.

Figure 1 shows a 3D frame, with beams of varying circular cross-sections, loaded by concentrated loads F at node 1. The displacements of nodes 1 to 4 were founded. Variation of the cross-sectional area of the beams a is defined by the following diameter quadratic function $d(y) = 0.04 + 0.04y^2$. The beams b and c have constant diameters through lengths of elements. Detailed parameters can be found in (Murín et al. 2002). Only one exact beam element was used to model each beam (a, b, c). In the Hermite beam element model, only one element was used to represent the beams b and c in all cases, but beams a were modelled with 1, 2 and 3 elements in models 1, 2 and 3 respectively.

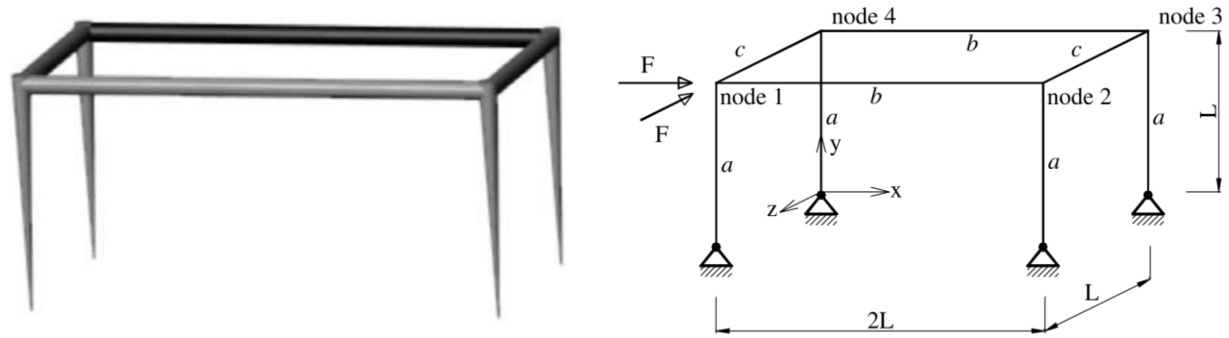

Figure 1. Frame displacement at node 13 **(Murín et al. 2002)**

The numerical results obtained by the present method are compared against those from the method proposed by Murín et al. (2002) and the solutions from classical Hermite beam elements, as presented in **Table 1**. It can be observed from the table that compared to the reference method, the displacement solutions of the present method at all nodes and under all loading cases are consistently closer to the exact solution, demonstrating a significant enhancement in computational accuracy. Furthermore, when the number of elements is varied, the present method exhibits a narrower and more stable variation range in its solutions, highlighting its superior numerical robustness.

Table 1. Comparison of results

	Node1 (errors %)		Node2 (errors %)		Node3 (errors %)		Node4 (errors %)	
	U_x(mm)	U_z(mm)	U_x(mm)	U_z(mm)	U_x(mm)	U_z(mm)	U_x(mm)	U_z(mm)
Exact solution	0.775	-1.098	0.774	-0.428	0.945	-0.428	0.945	-1.098
Model1 ref	0.651(16.0)	-0.882(19.7)	0.650	-0.336(21.5)	0.763(19.3)	-0.336	0.763	-0.882
Model1 this paper	0.745(3.9)	-0.981(10.7)	0.745	-0.427(0.2)	0.859(9.1)	-0.427	0.859	-0.981
Model2 ref	0.743(4.1)	-1.008(8.2)	0.722	-0.390(8.9)	0.869(8.0)	-0.390	0.869	-1.008
Model2 this paper	0.767(1.0)	-1.085(1.2)	0.766	-0.423(1.2)	0.933(1.3)	-0.423	0.933	-1.086
Model3 ref	0.749(3.4)	-1.054(4.0)	0.748	-0.409(4.4)	0.908(3.9)	-0.409	0.908	-1.054
Model3 this paper	0.772(0.4)	-1.093(0.5)	0.771	-0.426(0.5)	0.940(0.5)	-0.426	0.940	-1.093

- The presentation should be improved. Section 3 presents the co-rotational framework, which is well-known in the literature, but no references are cited. A 3D beam is considered, but the equilibrium equation (1) is written for a 2D beam. An explanation for Eqs. (1)-(3) should be given.

Reply:

Thank you for your valuable suggestions. Citations have been added to the co-rotational formulation derivation section in the manuscript as recommended, with the corresponding references provided below. Regarding the three-dimensional beam problem, the expressions for the y-direction and z-direction in Equations (1-3) are analogous. A supplementary explanation has been incorporated into the relevant section of the main text.

Ref:

[1] Crisfield MA.: A consistent co-rotational formulation for non-linear, three-dimensional, beam-elements[J]. Comput Method Appl M 1990;81(2):131-150.

[2] Crisfield MA, Moita GF.: A unified co-rotational framework for solids, shells and beams[J]. Int J Solids Struct 1996;33(20-22):2969-2992.

- More information on the large deformation behavior of the structure, such as snap-through and snap-back, is required to show the efficiency of the element and numerical algorithm.

Reply:

Thank you for your suggestion. The issue you raised is of significant research value. Analyzing complex post-buckling behaviors involving snap-through and snap-back indeed serves as a rigorous benchmark for evaluating the robustness of nonlinear beam elements and algorithms.

In this paper, our primary objective is to develop an efficient co-rotational formulation for three-dimensional variable cross-section Timoshenko beams and to validate its accuracy and convergence on a series of fundamental yet critical large-displacement and large-rotation problems. We believe these results establish a reliable foundation for the proposed method.

The type of strongly nonlinear problems involving limit points and unstable paths that you highlighted indeed represents a very important and natural extension of this method's application. As demonstrated in the reference Battini, J.-M. (2008), similar problems have also been analyzed using the co-rotational method. We also plan to dedicate future work to specifically investigate and demonstrate the application of this method for full-path tracking of post-buckling behavior in structures such as arches and domes. Once again, we appreciate this constructive suggestion, which has helped clarify the key directions for our future research.

Ref:

[3] Battini, Jean-Marc. "A non-linear corotational 4-node plane element." Mechanics research communications 35.6 (2008): 408-413.