



Scale-Dependent Anisotropy in Two-Dimensional Spectra of Wind-Turbine Wake Boundary-Layer Flows

Sedigheh Sadeghi¹, Mahdi Ghane¹, and Naseem Ali²

¹Department of Computer Science, Electrical Engineering and Mathematical Sciences, Western Norway University of Applied Sciences, 5528 Haugesund, Norway

²Institute of Coastal Systems – Analysis and Modeling, Helmholtz-Zentrum Hereon, Max-Planck-Str. 1, 21502 Geesthacht, Germany

Correspondence: Mahdi Ghane (Mahdi.Ghane@hvl.no)

Abstract. Accurate characterization of anisotropic turbulence in wind-turbine wakes requires spectral models that retain both directional information and its variation across scales. Although one-dimensional spectra are widely used to compare measured and simulated wake turbulence, they collapse the underlying wavevector structure and therefore cannot fully describe the geometry of anisotropic energy organization. We propose a two-dimensional spectral framework based on a bidirectional, scale-dependent tensorial remapping of the planar wavenumber domain. The model represents the strong anisotropy observed at energy-containing scales and its gradual relaxation toward smaller-scale isotropic behavior. This transition is described by two directional stretch factors that depend jointly on the streamwise and spanwise wavenumbers. The corresponding Jacobian correction ensures that the spectral density is transformed consistently under wavevector mapping. The scale dependence of the directional stretching is represented by using constrained Bernstein polynomial surfaces. A cumulative softplus parameterization guarantees boundedness, smoothness, and monotonic decay throughout the optimization. The model is applied independently to large-eddy simulation datasets representing wind-turbine wakes under high- and low-turbulence inflow conditions, together with a reference atmospheric boundary-layer case without turbine forcing. Parameters are estimated using a staged, coarse-to-fine constrained optimization of the planar spectra of the streamwise and wall-normal velocity components. Component-energy constraints, Jacobian admissibility, and surface-smoothness regularization are included to maintain physical consistency and numerical robustness. The fitted spectra are also used to generate statistically equivalent synthetic turbulent fields to evaluate spatial correlations, integral scales, structure functions, and one-point statistics. The resulting framework provides an interpretable and computationally practical method for quantifying how turbine wakes and inflow turbulence intensity modify the directional, scale-dependent organization of turbulent energy.

1 Introduction

Understanding the complex behavior of wind-turbine wakes is essential for improving the performance and productivity of wind farms. Wake development governs power losses, turbine–turbine interactions, structural loading, and the rate at which momentum is transported from the atmospheric boundary layer (ABL) into the wake (Cal et al., 2010). These processes are strongly influenced by atmospheric turbulence, which is intrinsically anisotropic, inhomogeneous, and scale dependent. In



turbine-affected flows, the organization of turbulence is further modified by rotor induction, wake rotation, shear production, tip-vortex dynamics, wake meandering, and stability-dependent recovery. Consequently, a physically meaningful description of turbulence must account not only for its intensity, but also for how turbulent energy is distributed among velocity components, spatial directions, and scales (Ali et al., 2016, 2019).

Spectral analysis provides a natural framework for this purpose because it connects turbulent energy with characteristic length scales, coherent structures, and dynamical processes across wavenumber or frequency space (Favre, 1965). Atmospheric wind variability extends over a broad range of scales, from synoptic and mesoscale motions to microscale turbulence. Early observations established that different scale ranges can exhibit distinct power-law behavior (Vinnichenko, 1970). From a theoretical perspective, two-dimensional turbulence admits an inverse energy cascade associated with a $k^{-5/3}$ spectrum, where k denotes wavenumber, and a forward enstrophy cascade associated with a k^{-3} spectrum (Kraichnan, 1967). However, Lindborg (1999) showed that, although the k^{-3} range may be interpreted consistently in terms of a two-dimensional enstrophy cascade, the atmospheric $k^{-5/3}$ range cannot generally be explained as a two-dimensional inverse energy cascade. Related scale transitions have also been observed in offshore and boundary-layer wind records. Measurements over water indicate that mesoscale wind spectra may follow an approximately $f^{-5/3}$ scaling, where f denotes frequency, before transitioning toward f^{-3} behavior at synoptic and planetary scales (Larsén et al., 2013). Cheynet et al. (2018) and Larsén et al. (2016) demonstrated the usefulness of spectral models for identifying characteristic slopes, transitions, and possible spectral gaps across atmospheric scales (Syed and Mann, 2024a).

Most atmospheric and wind-energy spectral analyses nevertheless rely on one-dimensional spectra obtained from temporal measurements, often through Taylor's frozen-turbulence hypothesis (Tennekes and Lumley, 1972; Ali and Cal, 2019). Although convenient, a one-dimensional spectrum collapses the underlying multidimensional wavevector field onto a single coordinate. Contributions from motions with different lateral or vertical wavenumbers are therefore combined within the same streamwise spectrum. This projection can obscure spectral peaks, integral scales, and the geometric organization of energy-containing structures, particularly at low streamwise wavenumbers (Kelly and Wyngaard, 2006). One-dimensional power-law behavior may also result from the integration of a more complex two-dimensional spectral distribution and does not, by itself, uniquely identify the geometry or dynamics of the underlying turbulent structures (Chung et al., 2015). Planar two-dimensional spectra provide additional information by retaining the joint dependence of spectral energy on two wavevector components and thereby distinguishing streamwise organization from transverse or wall-normal structure. The need for multidimensional spectral descriptions is especially pronounced in wind-turbine wakes. Turbine-induced turbulence is not simply an amplified form of the incoming ABL turbulence. Rotor loading, wake rotation, velocity-deficit shear, coherent tip vortices, and wake meandering redistribute energy across both scales and directions. The induction region may modify low-wavenumber fluctuations, while downstream shear-layer growth and enhanced mixing progressively alter the spatial organization of the wake (Placidi et al., 2023). Wake recovery and coherent-structure evolution also depend strongly on ambient turbulence intensity, atmospheric stability, wind shear, veer, and inflow organization (Gouder et al., 2024; Kosović et al., 2026). A streamwise-only spectral representation cannot determine whether enhanced low-wavenumber energy originates from long streamwise structures, transverse wake motion, wake meandering, or projection of cross-flow fluctuations.



Modal analyses provide further evidence that wake dynamics are intrinsically multidirectional. Spectral proper orthogonal decomposition and related coherent-structure methods have identified coupled streamwise, radial, and azimuthal motions in turbine wakes, particularly under stability-dependent conditions and active wake control (Yalla et al., 2025). These structures cannot be fully characterized through one-dimensional spectra alone. At the same time, measurements that resolve turbulent fluctuations simultaneously in space and time remain limited in field-scale wind-energy applications (de Kat and Ganapathisubramani, 2015). Practical models must therefore balance two competing requirements that they should be sufficiently compact for use with limited observations and reduced-order descriptions, while retaining enough directional information to represent the scale-dependent geometry of anisotropic turbulence (Syed and Mann, 2024a). Recent two-dimensional spectral-tensor models for marine-atmospheric winds and spectral reconstruction of neutral ABL turbulence demonstrate that this balance is achievable (Syed and Mann, 2024b; Alreweny et al., 2024).

The interpretation of scale-dependent anisotropy is also connected to long-standing questions concerning the organization of wall-bounded turbulence. Attached-eddy theories predict self-similar populations of energy-containing structures and characteristic inertial-range scaling in the logarithmic region (Townsend, 1976; Perry and Chong, 1982; Hunt and Morrison, 2000). Experimental and particle-image-velocimetry studies have reported evidence of structural self-similarity (Morrison and Kronauer, 1969; Tomkins and Adrian, 2005), whereas other investigations suggest that the streamwise length and transverse width of energetic structures may not obey strict geometric similarity (Del Álamo and Jiménez, 2009). Part of this ambiguity arises because one-dimensional spectra can mix contributions from structures with different aspect ratios and orientations (Rosenberg et al., 2013; Davidson et al., 2006). These issues become even more important in turbine wakes, where rotor-induced shear, rotation, and meandering may modify both the scale and directional organization of the energetic motions. Despite these considerations, planar two-dimensional spectra remain comparatively underused in wall-bounded turbulence and turbine-wake analysis (Chandran et al., 2017). Moreover, many existing anisotropic spectral models employ a constant directional stretching factor, implicitly assuming that the same anisotropy applies across the entire wavenumber domain. Such a representation is restrictive because large energy-containing motions may remain strongly anisotropic while smaller scales progressively approach a more isotropic state. The rate of this relaxation may also differ between the streamwise and spanwise spectral directions.

A suitable model must therefore permit the anisotropy to evolve across the full planar wavevector domain while preserving spectral density and maintaining a physically admissible coordinate transformation. The present study addresses this gap by introducing a bidirectional, scale-dependent tensorial remapping for planar two-dimensional spectra of turbine-affected ABL turbulence. The principal contributions of the study are fourfold. First, we formulate a two-dimensional wavevector remapping that allows anisotropy to vary jointly along both spectral directions rather than through a single constant stretching parameter. Second, we introduce a physically constrained Bernstein representation that guarantees admissible, smooth, and monotonically decaying anisotropy surfaces throughout the optimization. Third, the model parameters are estimated using a staged, coarse-to-fine nonlinear optimization and surface-smoothness regularization. Finally, the fitted spectra are used to generate statistically equivalent synthetic turbulent fields, enabling additional assessment through one-point statistics.

The remainder of the paper is organized as follows. Section 2 introduces the two-dimensional wavevector-anisotropy formulation and the scale-dependent spectral remapping. Section 3 presents the parameter-estimation and optimization strategy.



Section 4 describes the generation and validation of synthetic turbulent fields from the fitted spectra. Section 5 presents the
 95 LES cases, turbine configuration, inflow conditions, and relevant flow parameters. Section 6 evaluates the ability of the model
 to capture directional energy organization, scale-dependent anisotropy. Finally, Section 7 summarizes the main conclusions
 and outlines directions for future work.

2 Theory: Two-dimensional wavenumber spectrum model

For a statistically homogeneous flow, let $u'_i(\mathbf{x})$ denote velocity fluctuations and the two-point correlation tensor presented as
 100 (Pope, 2000)

$$R_{ij}(\mathbf{r}) = \langle u'_i(\mathbf{x}) u'_j(\mathbf{x} + \mathbf{r}) \rangle, \quad (1)$$

The three-dimensional (3-D) spectral density tensor $\Phi_{ij}(\mathbf{k})$ is defined by the Fourier pair as

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} R_{ij}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3\mathbf{r}, \quad R_{ij}(\mathbf{r}) = \int_{\mathbb{R}^3} \Phi_{ij}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3\mathbf{k}. \quad (2)$$

The two-dimensional (2-D) spectral tensor on a plane spanned by wavenumber coordinates k_p and k_q is obtained by integrating
 105 the 3-D tensor over the remaining coordinate k_s as

$$\Phi_{mn}^{2D}(k_p, k_q) = \int_{-\infty}^{\infty} \Phi_{mn}(k_p, k_q, k_s) dk_s, \quad \{p, q, s\} = \{1, 2, 3\}, \quad m, n \in \{1, 2, 3\}. \quad (3)$$

Summing the diagonal yields the 2-D energy spectrum on that plane as

$$E^{2D}(k_p, k_q) = \frac{1}{2} \sum_{m=1}^3 \Phi_{mm}^{2D}(k_p, k_q), \quad (4)$$

which conserves total energy under planar integration:

$$110 \quad \int_{\mathbb{R}^2} E^{2D}(k_p, k_q) dk_p dk_q = \frac{1}{2} \sum_{m=1}^3 \langle u_m'^2 \rangle. \quad (5)$$

When an in-plane radial representation is useful, define $k_r = \sqrt{k_p^2 + k_q^2}$ and the azimuthally averaged 2-D spectrum as

$$\bar{E}_r(k_r) = \frac{1}{2\pi} \int_0^{2\pi} E^{2D}(k_r, \varphi) d\varphi, \quad (6)$$

where φ is the polar angle in the spectral plane. For anisotropic flows, it is preferable to work with the component-wise planar
 variance spectra

$$115 \quad E_{mm}^{2D}(k_p, k_q) = \Phi_{mm}^{2D}(k_p, k_q), \quad (7)$$



which satisfy

$$\int_{\mathbb{R}^2} E_{mm}^{2D}(k_p, k_q) dk_p dk_q = \langle u_m'^2 \rangle. \quad (8)$$

These component spectra retain directional information and form the basis for the elliptic, scale-dependent anisotropy modeling used later. A practical way to characterize wavevector anisotropy is to introduce a coordinate transform that maps the measured wavevector coordinates to a baseline coordinate system. Because anisotropy is unlikely to be uniform across the inertial subrange—from the outer to the inner scales—we introduce direction-specific, scale-dependent stretch factors $\mu_\ell(\mathbf{k})$ that measure the departure from the no-remapping limit.

Studies on atmospheric turbulence have revealed distinct behaviors of streamwise velocity at various scales Cheng et al. (2020). These observations indicate that the statistical properties of eddies vary with direction and thereby violate the axisymmetric assumption of the spectrum of the horizontal plane. To investigate the characteristics of convective boundary layer flows more comprehensively, it is necessary to develop anisotropic models for the wavenumber spectrum. By adopting the notion of directional anisotropy, it is possible to develop a quantitative model for the anisotropic spectrum that pertains to the streamwise velocity. The directional anisotropy can be determined by means of a scale-by-scale analysis of the ratio of the second-order moments (Narita, 2017). The spectral moment method allows for the automatic determination of the ratio of the major to minor axis anisotropy through consideration of the second-order moment. In contrast to conventional anisotropic models based on a single constant stretching factor, we introduce a bidirectional scale-dependent tensorial remapping in spectral space. The central idea is that turbulence is strongly anisotropic at large scales and gradually approaches isotropy toward smaller scales, but this relaxation need not occur only along the streamwise direction. Instead, the anisotropy may vary across the full planar spectral domain, and therefore the remapping must depend on both wavenumber components. Let the planar wavenumber vector be

$$\mathbf{k} = (k_1, k_2)^\top, \quad (9)$$

where $k_1 = k_x$ denotes the streamwise wavenumber and $k_2 = k_y$ the spanwise wavenumber on the horizontal measurement plane. We define a transformed spectral coordinate

$$\boldsymbol{\kappa} = \mathbf{T}(\mathbf{k}), \quad (10)$$

such that the measured anisotropic spectrum is represented as a remapped version of baseline component spectra in transformed coordinates, derived from an isotropic spectral-tensor form. To describe the gradual return to isotropy along both spectral directions, we introduce the normalized logarithmic scale variables as

$$\hat{q}_i(k_i) = \begin{cases} 0, & 0 < k_i \leq k_{f,i}, \\ \frac{\ln(k_i/k_{f,i})}{\ln(k_{s,i}/k_{f,i})}, & k_{f,i} < k_i < k_{s,i}, \\ 1, & k_i \geq k_{s,i}, \end{cases} \quad i \in \{1, 2\}, \quad (11)$$



where $k_{f,i} = 2\pi/L_{f,i}$ denotes the forcing wavenumber associated with the large energy-containing scale in direction i , and
 145 $k_{s,i}$ is the wavenumber beyond which the turbulence is assumed to be effectively isotropic in that direction. Thus, $\hat{q}_i = 0$
 corresponds to the largest dynamically active scales, while $\hat{q}_i = 1$ corresponds to the onset of effective isotropy. For each
 principal direction $\ell \in \{1, 2\}$, we define the anisotropy factor by

$$\mu_\ell(\mathbf{k}) = 1 + (\mu_{\ell,\max} - 1) \Psi_\ell(\hat{q}_1(k_1), \hat{q}_2(k_2)), \quad (12)$$

where $\mu_{\ell,\max} > 1$ denotes the large-scale anisotropy level in principal direction ℓ , and $\Psi_\ell : [0, 1]^2 \rightarrow [0, 1]$ is a bivariate decay
 150 function satisfying

$$\frac{\partial \Psi_\ell}{\partial q_1} \leq 0, \quad \frac{\partial \Psi_\ell}{\partial q_2} \leq 0, \quad (q_1, q_2) \in [0, 1]^2. \quad (13)$$

Hence, the anisotropy is strongest at the large-scale corner $(q_1, q_2) = (0, 0)$ and decreases monotonically as either coordinate
 moves toward the small-scale regime. The full isotropic no-remapping limit is attained at the corner $(q_1, q_2) = (1, 1)$, where
 $\mu_1 = \mu_2 = 1$, without requiring the anisotropy to vanish identically along the edges $q_1 = 1$ or $q_2 = 1$. Thus, residual anisotropy
 155 may persist when one spectral direction has reached the small-scale regime while the other has not. We represent Ψ_ℓ using a
 low-order bivariate Bernstein polynomial family,

$$\Psi_\ell(q_1, q_2) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} c_{ij}^{(\ell)} B_i^{(N_1)}(q_1) B_j^{(N_2)}(q_2), \quad (14)$$

where

$$B_i^{(N_1)}(q_1) = \binom{N_1}{i} q_1^i (1 - q_1)^{N_1-i}, \quad B_j^{(N_2)}(q_2) = \binom{N_2}{j} q_2^j (1 - q_2)^{N_2-j}. \quad (15)$$

160 To guarantee physical admissibility, we impose the coefficient constraints

$$c_{00}^{(\ell)} = 1, \quad c_{N_1, N_2}^{(\ell)} = 0, \quad (16)$$

and

$$0 \leq c_{ij}^{(\ell)} \leq 1, \quad c_{i+1,j}^{(\ell)} \leq c_{ij}^{(\ell)}, \quad c_{i,j+1}^{(\ell)} \leq c_{ij}^{(\ell)}. \quad (17)$$

These conditions ensure that Ψ_ℓ decays monotonically in both spectral directions while satisfying the corner conditions
 165 $\Psi_\ell(0, 0) = 1$ and $\Psi_\ell(1, 1) = 0$. In contrast to the more restrictive edge-vanishing formulation, this allows residual anisotropy
 when one spectral direction has reached the small-scale regime while the other has not. The partial derivatives of Ψ_ℓ are

$$\frac{\partial \Psi_\ell}{\partial q_1} = N_1 \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2} (c_{i+1,j}^{(\ell)} - c_{ij}^{(\ell)}) B_i^{(N_1-1)}(q_1) B_j^{(N_2)}(q_2), \quad (18)$$

$$\frac{\partial \Psi_\ell}{\partial q_2} = N_2 \sum_{i=0}^{N_1} \sum_{j=0}^{N_2-1} (c_{i,j+1}^{(\ell)} - c_{ij}^{(\ell)}) B_i^{(N_1)}(q_1) B_j^{(N_2-1)}(q_2). \quad (19)$$



170 Because the Bernstein basis functions are non-negative and the coefficient increments are non-positive by Eq. (17), it follows that

$$\frac{\partial \Psi_\ell}{\partial q_1} \leq 0, \quad \frac{\partial \Psi_\ell}{\partial q_2} \leq 0. \quad (20)$$

The derivatives of the normalized logarithmic scale variables are

$$\frac{\partial \hat{q}_i}{\partial k_i} = \begin{cases} 0, & 0 < k_i \leq k_{f,i}, \\ \frac{1}{k_i \ln(k_{s,i}/k_{f,i})}, & k_{f,i} < k_i < k_{s,i}, \\ 0, & k_i \geq k_{s,i}, \end{cases} \quad i \in \{1, 2\}. \quad (21)$$

175 Therefore, by the chain rule,

$$\frac{\partial \mu_\ell}{\partial k_1} = (\mu_{\ell, \max} - 1) \frac{\partial \Psi_\ell}{\partial q_1}(\hat{q}_1(k_1), \hat{q}_2(k_2)) \frac{\partial \hat{q}_1}{\partial k_1}, \quad (22)$$

$$\frac{\partial \mu_\ell}{\partial k_2} = (\mu_{\ell, \max} - 1) \frac{\partial \Psi_\ell}{\partial q_2}(\hat{q}_1(k_1), \hat{q}_2(k_2)) \frac{\partial \hat{q}_2}{\partial k_2}. \quad (23)$$

Within the transition region $k_{f,1} < k_1 < k_{s,1}$ and $k_{f,2} < k_2 < k_{s,2}$, these reduce to

180
$$\frac{\partial \mu_\ell}{\partial k_1} = \frac{\mu_{\ell, \max} - 1}{k_1 \ln(k_{s,1}/k_{f,1})} \frac{\partial \Psi_\ell}{\partial q_1}, \quad (24)$$

$$\frac{\partial \mu_\ell}{\partial k_2} = \frac{\mu_{\ell, \max} - 1}{k_2 \ln(k_{s,2}/k_{f,2})} \frac{\partial \Psi_\ell}{\partial q_2}. \quad (25)$$

Hence,

$$\frac{\partial \mu_\ell}{\partial k_1} \leq 0, \quad \frac{\partial \mu_\ell}{\partial k_2} \leq 0, \quad (26)$$

185 which means that anisotropy weakens monotonically as either spectral coordinate increases. The anisotropy is described by the positive-definite stretch tensor

$$\mathbf{A}(\mathbf{k}) = \mathbf{R}(\theta) \begin{pmatrix} \mu_1(\mathbf{k}) & 0 \\ 0 & \mu_2(\mathbf{k}) \end{pmatrix} \mathbf{R}^\top(\theta), \quad (27)$$

where $\mathbf{R}(\theta)$ is the planar rotation matrix. If the principal axes of anisotropy are aligned with the (k_1, k_2) axes, then $\theta = 0$ and Eq. (27) reduces to

190
$$\mathbf{A}(\mathbf{k}) = \begin{pmatrix} \mu_1(\mathbf{k}) & 0 \\ 0 & \mu_2(\mathbf{k}) \end{pmatrix}. \quad (28)$$



In the applications considered here, the spectral axes are taken to coincide with the streamwise and spanwise directions. Accordingly, $\theta = 0$ is fixed rather than estimated; the rotated form in Eq. (27) is retained as a general extension for datasets with tilted spectral principal axes. The associated metric tensor is

$$\mathbf{G}(\mathbf{k}) = \mathbf{A}^\top(\mathbf{k})\mathbf{A}(\mathbf{k}), \quad (29)$$

195 which is symmetric positive-definite by construction. In the aligned case, the metric tensor is written as

$$\mathbf{G}(\mathbf{k}) = \begin{pmatrix} \mu_1^2(\mathbf{k}) & 0 \\ 0 & \mu_2^2(\mathbf{k}) \end{pmatrix}. \quad (30)$$

The directional stretch formulation above can be rewritten in invariant form by separating isotropic dilation from anisotropic shape. Define

$$a(\mathbf{k}) = \sqrt{\mu_1(\mathbf{k})\mu_2(\mathbf{k})}, \quad \gamma(\mathbf{k}) = \frac{1}{2} \ln \left(\frac{\mu_1(\mathbf{k})}{\mu_2(\mathbf{k})} \right). \quad (31)$$

200 Then

$$\mu_1(\mathbf{k}) = a(\mathbf{k}) e^{\gamma(\mathbf{k})}, \quad \mu_2(\mathbf{k}) = a(\mathbf{k}) e^{-\gamma(\mathbf{k})}, \quad (32)$$

and the stretch tensor may be written equivalently as

$$\mathbf{A}(\mathbf{k}) = a(\mathbf{k}) \mathbf{R}(\theta) \begin{pmatrix} e^{\gamma(\mathbf{k})} & 0 \\ 0 & e^{-\gamma(\mathbf{k})} \end{pmatrix} \mathbf{R}^\top(\theta). \quad (33)$$

This decomposition shows that the original remapping contains three distinct ingredients, including an isotropic dilation field
 205 $a(\mathbf{k})$, an anisotropy magnitude field $\gamma(\mathbf{k})$, and an orientation angle θ . Substituting Eq. (33) into Eq. (29) yields

$$\mathbf{G}(\mathbf{k}) = a^2(\mathbf{k}) \mathbf{R}(\theta) \begin{pmatrix} e^{2\gamma(\mathbf{k})} & 0 \\ 0 & e^{-2\gamma(\mathbf{k})} \end{pmatrix} \mathbf{R}^\top(\theta). \quad (34)$$

Normalizing by the determinant removes the isotropic scale and isolates the pure anisotropic shape as

$$\mathbf{B}(\mathbf{k}) = \frac{\mathbf{G}(\mathbf{k})}{\det(\mathbf{G}(\mathbf{k}))^{1/2}} = \mathbf{R}(\theta) \begin{pmatrix} e^{2\gamma(\mathbf{k})} & 0 \\ 0 & e^{-2\gamma(\mathbf{k})} \end{pmatrix} \mathbf{R}^\top(\theta). \quad (35)$$

Thus, $\gamma(\mathbf{k})$ provides an invariant measure of anisotropy, while $a(\mathbf{k})$ captures the isotropic part of the remapping. An invariant
 210 scalar anisotropy measure is defined as

$$I_2(\mathbf{k}) = \frac{1}{2} \text{tr} \left[(\log \mathbf{B}(\mathbf{k}))^2 \right] = 4\gamma^2(\mathbf{k}). \quad (36)$$

Hence, $\gamma(\mathbf{k}) = 0$, or equivalently $\mu_1(\mathbf{k}) = \mu_2(\mathbf{k})$, corresponds to an isotropic shape of the remapping. The full isotropic no-remapping limit is recovered only when, in addition, $a(\mathbf{k}) = 1$, so that $\mu_1(\mathbf{k}) = \mu_2(\mathbf{k}) = 1$. We then define the transformed wavevector by



$$\kappa(\mathbf{k}) = \mathbf{A}(\mathbf{k}) \mathbf{k}. \quad (37)$$

In the aligned case, the components of the wavevector is presented as

$$\kappa_1 = \mu_1(\mathbf{k}) k_1, \quad \kappa_2 = \mu_2(\mathbf{k}) k_2. \quad (38)$$

Thus, the transformed magnitude becomes

$$\kappa = \|\kappa\| = \sqrt{\mathbf{k}^\top \mathbf{G}(\mathbf{k}) \mathbf{k}} = \sqrt{\mu_1^2(\mathbf{k}) k_1^2 + \mu_2^2(\mathbf{k}) k_2^2}. \quad (39)$$

220 Because the remapping depends on both k_1 and k_2 , the Jacobian of the transformation must be computed with respect to the full planar wavevector. In compact form,

$$\nabla_{\mathbf{k}} \mathbf{T}(\mathbf{k}) = \mathbf{A}(\mathbf{k}) + \left[\frac{\partial \mathbf{A}}{\partial k_1} \mathbf{k}, \frac{\partial \mathbf{A}}{\partial k_2} \mathbf{k} \right], \quad (40)$$

and therefore

$$J_s(\mathbf{k}) = \det(\nabla_{\mathbf{k}} \mathbf{T}(\mathbf{k})), \quad (41)$$

225 where J_s denotes the signed determinant. In the aligned case $\theta = 0$, the Jacobian matrix becomes

$$\frac{\partial(\kappa_1, \kappa_2)}{\partial(k_1, k_2)} = \begin{pmatrix} \mu_1(\mathbf{k}) + k_1 \frac{\partial \mu_1}{\partial k_1} & k_1 \frac{\partial \mu_1}{\partial k_2} \\ k_2 \frac{\partial \mu_2}{\partial k_1} & \mu_2(\mathbf{k}) + k_2 \frac{\partial \mu_2}{\partial k_2} \end{pmatrix}, \quad (42)$$

so that

$$J_s(k_1, k_2) = \left(\mu_1 + k_1 \frac{\partial \mu_1}{\partial k_1} \right) \left(\mu_2 + k_2 \frac{\partial \mu_2}{\partial k_2} \right) - k_1 k_2 \frac{\partial \mu_1}{\partial k_2} \frac{\partial \mu_2}{\partial k_1}. \quad (43)$$

The non-negative density-transformation factor used in the modeled spectra is

$$230 \quad J(k_1, k_2) = |J_s(k_1, k_2)|. \quad (44)$$

This is the key extension relative to the one-dimensional model that the Jacobian is no longer triangular, and a genuine cross-coupling term appears through the mixed directional dependence of the anisotropy factors. The remapping $\mathbf{k} \mapsto \kappa(\mathbf{k})$ is locally one-to-one and orientation-preserving under the following conditions

$$\mu_1(\mathbf{k}) > 0, \quad \mu_2(\mathbf{k}) > 0, \quad \det(\nabla_{\mathbf{k}} \mathbf{T}(\mathbf{k})) > 0 \quad (45)$$



235 for all $\mathbf{k}$ in the fitting domain. In the aligned case, a sufficient set of conditions is

$$\mu_1 + k_1 \frac{\partial \mu_1}{\partial k_1} > 0, \quad \mu_2 + k_2 \frac{\partial \mu_2}{\partial k_2} > 0, \quad (46)$$

together with

$$\left(\mu_1 + k_1 \frac{\partial \mu_1}{\partial k_1} \right) \left(\mu_2 + k_2 \frac{\partial \mu_2}{\partial k_2} \right) > k_1 k_2 \frac{\partial \mu_1}{\partial k_2} \frac{\partial \mu_2}{\partial k_1}. \quad (47)$$

240 These conditions prevent folding of the spectral map. Now, let $\tilde{E}_u(\boldsymbol{\kappa})$ and $\tilde{E}_w(\boldsymbol{\kappa})$ denote baseline component spectra in the transformed coordinates. We adopt von Kármán-type component forms, derived from an isotropic spectral-tensor baseline and supplemented by a dissipation-range correction, as (Glegg and Devenport, 2017)

$$\tilde{E}_u(\boldsymbol{\kappa}) = \frac{A_u}{18\pi} \frac{\overline{u'^2}}{k_e^2} \frac{3 + 3(\kappa_1/k_e)^2 + 11(\kappa_2/k_e)^2}{[1 + (\kappa/k_e)^2]^{7/3+\beta}} D(\kappa), \quad (48)$$

$$\tilde{E}_w(\boldsymbol{\kappa}) = \frac{4A_w}{9\pi} \frac{\overline{w'^2}}{k_e^2} \frac{(\kappa/k_e)^2}{[1 + (\kappa/k_e)^2]^{7/3+\beta}} D(\kappa), \quad (49)$$

245 where

$$D(\kappa) = \exp \left\{ -\xi \left([(\kappa\eta)^4 + c_\eta^4]^{1/4} - c_\eta \right) \right\}. \quad (50)$$

For admissible remappings satisfying $J_s(\mathbf{k}) > 0$, the anisotropic planar spectra are defined by evaluating the baseline component spectra at the transformed coordinates and multiplying by the non-negative density factor $J(\mathbf{k}) = |J_s(\mathbf{k})|$ as

$$E_u^M(k_1, k_2) = \tilde{E}_u(\boldsymbol{\kappa}(k_1, k_2)) J(k_1, k_2), \quad (51)$$

250

$$E_w^M(k_1, k_2) = \tilde{E}_w(\boldsymbol{\kappa}(k_1, k_2)) J(k_1, k_2). \quad (52)$$

Eqs. (51)–(52) preserve the full bidirectional scale dependence of the anisotropy inside the spectrum and eliminate the need for any centroid-based reduction to a single effective parameter. Let E_u^D and E_w^D denote the measured planar spectra, and E_u^M , E_w^M the corresponding model spectra. For the implementation used in this study, the active parameter vector is

$$255 \quad \Theta = \left\{ A_u, A_w, k_e, \beta, \eta, \xi, c_\eta, \mu_{1,\max}, \mu_{2,\max}, \boldsymbol{\vartheta}^{(1)}, \boldsymbol{\vartheta}^{(2)}, k_{f,1}, k_{s,1}, k_{f,2}, k_{s,2} \right\}, \quad (53)$$

where the admissible Bernstein matrices are derived as $\mathbf{C}^{(\ell)} = \mathcal{C}(\boldsymbol{\vartheta}^{(\ell)})$ from unconstrained raw variables through the cumulative-softplus construction described below. The rotation angle is fixed at $\theta = 0$ for all reported fits. We define the joint objective function

$$\mathcal{L}(\Theta) = \lambda_u \mathcal{L}_u + \lambda_w \mathcal{L}_w + \lambda_E \mathcal{L}_E + \lambda_J \mathcal{L}_J + \lambda_S \mathcal{L}_S. \quad (54)$$



260 The spectral mismatch terms are measured in logarithmic space as

$$\mathcal{L}_u = \sum_{m=1}^{N_{k_1}} \sum_{n=1}^{N_{k_2}} W_{mn} [\log(E_u^D(k_{1,m}, k_{2,n}) + \varepsilon) - \log(E_u^M(k_{1,m}, k_{2,n}) + \varepsilon)]^2, \quad (55)$$

$$\mathcal{L}_w = \sum_{m=1}^{N_{k_1}} \sum_{n=1}^{N_{k_2}} W_{mn} [\log(E_w^D(k_{1,m}, k_{2,n}) + \varepsilon) - \log(E_w^M(k_{1,m}, k_{2,n}) + \varepsilon)]^2. \quad (56)$$

Here, $\varepsilon > 0$ is a small regularization constant and W_{mn} are optional spectral weights. In the implementation, the energy targets
 265 are the discrete integrals of the measured, windowed spectra,

$$\sigma_{u,D}^2 = \sum_{m,n} E_u^D(k_{1,m}, k_{2,n}) \Delta k_1 \Delta k_2, \quad \sigma_{w,D}^2 = \sum_{m,n} E_w^D(k_{1,m}, k_{2,n}) \Delta k_1 \Delta k_2. \quad (57)$$

To enforce consistency with these measured spectral variances, we add the energy constraint

$$\mathcal{L}_E = \left[\frac{\sum_{m,n} E_u^M(k_{1,m}, k_{2,n}) \Delta k_1 \Delta k_2 - \sigma_{u,D}^2}{\sigma_{u,D}^2} \right]^2 + \left[\frac{\sum_{m,n} E_w^M(k_{1,m}, k_{2,n}) \Delta k_1 \Delta k_2 - \sigma_{w,D}^2}{\sigma_{w,D}^2} \right]^2. \quad (58)$$

The admissibility of the remapping is enforced through the Jacobian penalty as

$$270 \quad \mathcal{L}_J = \sum_{m,n} [\min(0, \det(\nabla_{\mathbf{k}} \mathbf{T}(\mathbf{k}_{mn})))^2]. \quad (59)$$

Here the penalty is applied to the signed determinant rather than to its absolute value, since local admissibility requires the remapping to be orientation-preserving. In the aligned case, Eq. (59) becomes

$$\mathcal{L}_J = \sum_{m,n} \left[\min \left(0, \left(\mu_1 + k_1 \frac{\partial \mu_1}{\partial k_1} \right) \left(\mu_2 + k_2 \frac{\partial \mu_2}{\partial k_2} \right) - k_1 k_2 \frac{\partial \mu_1}{\partial k_2} \frac{\partial \mu_2}{\partial k_1} \right) \right]^2. \quad (60)$$

To avoid overfitting of the anisotropy surface, we regularize the bivariate Bernstein coefficients using a two-dimensional
 275 smoothness penalty as

$$\mathcal{L}_S = \sum_{\ell=1}^2 \left(\mathcal{L}_{11}^{(\ell)} + \mathcal{L}_{22}^{(\ell)} + 2\mathcal{L}_{12}^{(\ell)} \right), \quad (61)$$

where

$$\mathcal{L}_{11}^{(\ell)} = \sum_{i=0}^{N_1-2} \sum_{j=0}^{N_2} \left(c_{i+2,j}^{(\ell)} - 2c_{i+1,j}^{(\ell)} + c_{i,j}^{(\ell)} \right)^2, \quad (62)$$

$$280 \quad \mathcal{L}_{22}^{(\ell)} = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2-2} \left(c_{i,j+2}^{(\ell)} - 2c_{i,j+1}^{(\ell)} + c_{i,j}^{(\ell)} \right)^2, \quad (63)$$



$$\mathcal{L}_{12}^{(\ell)} = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \left(c_{i+1,j+1}^{(\ell)} - c_{i+1,j}^{(\ell)} - c_{i,j+1}^{(\ell)} + c_{i,j}^{(\ell)} \right)^2. \quad (64)$$

The resulting optimization problem is defined as

$$\Theta^* = \arg \min_{\Theta} \mathcal{L}(\Theta), \quad (65)$$

285 subject to

$$A_u > 0, \quad A_w > 0, \quad k_e > 0, \quad \eta > 0, \quad \xi > 0, \quad c_\eta > 0, \quad \mu_{1,\max} > 1, \quad \mu_{2,\max} > 1, \quad (66)$$

and the coefficient constraints in Eqs. (16)–(17). In the numerical implementation, the coefficient inequalities in Eqs. (16)–(17) are not imposed directly on the optimizer. Instead, the Bernstein matrices are generated from unconstrained raw variables using a doubly cumulative softplus parameterization. This construction enforces the corner-only isotropic limit and coordinatewise
 290 monotonicity at every optimizer iteration while still allowing residual anisotropy along the edges $q_1 = 1$ or $q_2 = 1$.

3 Parameter estimation and optimization strategy

The proposed anisotropic spectral model is estimated independently for the WT-HTI, WT-LTI, and ABL datasets as a constrained nonlinear inverse problem. Let $E_u^D(k_1, k_2)$ and $E_w^D(k_1, k_2)$ denote the measured one-sided, quadrant-folded spectra of the streamwise and wall-normal velocity fluctuations, respectively. In the implementation, the first spatial coordinate is cropped
 295 to 120 points, the temporal mean is removed at every spatial location, and the data are permuted to the streamwise–spanwise plane. A periodic Hann window is then applied in both spatial directions, followed by a two-dimensional Fourier transform, exact folding of all reflected modes into the positive quadrant, window-power correction, time averaging, and normalization by $\Delta k_1 \Delta k_2$. The resulting spectra are therefore finite-window, locally homogeneous descriptions of the selected LES planes rather than spectra of a globally homogeneous wake. The corresponding model spectra are denoted by $E_u^M(k_1, k_2; \Theta)$ and
 300 $E_w^M(k_1, k_2; \Theta)$, where Θ is the vector of active model parameters. They are obtained by applying the bidirectional anisotropic remapping to baseline component spectra in the transformed coordinates and multiplying by $J = |J_s|$, so that the spectral density is transformed consistently in wavenumber space. The parameter vector is divided into physically interpretable blocks,

$$\Theta = \{\Theta_{\text{amp}}, \Theta_{\text{iso}}, \Theta_{\text{aniso}}, \Theta_{\text{geom}}\}, \quad (67)$$

305 where

$$\Theta_{\text{amp}} = \{A_u, A_w\}, \quad (68)$$

contains the component-wise amplitude parameters. The baseline component spectra are controlled by

$$\Theta_{\text{iso}} = \{k_e, \beta, \eta, \xi, c_\eta\}. \quad (69)$$



The anisotropic part of the model is controlled by

$$\Theta_{\text{aniso}} = \left\{ \mu_{1,\text{max}}, \mu_{2,\text{max}}, \boldsymbol{\vartheta}^{(1)}, \boldsymbol{\vartheta}^{(2)} \right\}, \quad (70)$$

where the Bernstein coefficient matrices are derived from the unconstrained raw vectors $\boldsymbol{\vartheta}^{(\ell)}$. Finally,

$$\Theta_{\text{geom}} = \{k_{f,1}, k_{s,1}, k_{f,2}, k_{s,2}\} \quad (71)$$

contains the active geometric parameters of the remapping. The rotation angle is fixed at $\theta = 0$ and is therefore not included in Θ . Thus, the block representation above is identical to the active parameter vector in Eq. (53). The same model structure, parameter bounds, and fitting workflow are applied separately to the three datasets.

The parameters are estimated by minimizing the composite objective function in Eq. (54). The spectral-shape mismatch is measured in logarithmic space because the spectra vary over several orders of magnitude. Forty logarithmic radial bands are assigned equal total weight, preventing the much larger number of high-wavenumber Cartesian cells from dominating the objective. The energy-consistency term constrains the modeled spectra to preserve the discrete spectral variances defined in Eq. (57). The Jacobian penalty enforces physical admissibility of the spectral remapping. A negative signed determinant J_s would indicate a locally folded or orientation-reversing transformation; therefore, only negative values of J_s are penalized as

$$\mathcal{L}_J = \sum_{m,n} [\min(J_{s,mn}, 0)]^2. \quad (72)$$

This term does not penalize large positive Jacobian directly. Instead, it specifically discourages nonphysical mappings that violate local orientation preservation. The final term regularizes the Bernstein coefficient surfaces that define the anisotropy-decay functions. For each principal direction $\ell \in \{1, 2\}$, second differences in the two coordinate directions and the mixed second difference are penalized as

$$\mathcal{L}_S = \sum_{\ell=1}^2 \left[\left\| \Delta_{11} \mathbf{C}^{(\ell)} \right\|_F^2 + \left\| \Delta_{22} \mathbf{C}^{(\ell)} \right\|_F^2 + 2 \left\| \Delta_{12} \mathbf{C}^{(\ell)} \right\|_F^2 \right]. \quad (73)$$

Here, $(\|\cdot\|_F)$ denotes the Frobenius norm. This smoothness penalty suppresses spurious oscillations in the fitted anisotropy surfaces and improves identifiability by discouraging overly irregular scale-dependent corrections that may fit noise rather than physical structure. All physical constraints are imposed through smooth reparameterizations rather than by directly imposing hard constraints on the numerical optimizer. Positive parameters are optimized in logarithmic form. For example, a positive scalar parameter $p > 0$ is represented as

$$p = \exp(x), \quad (74)$$

where $x \in \mathbb{R}$ is the unconstrained optimization variable. Bounded parameters are mapped through a sigmoid transformation,

$$p = p_{\min} + \frac{p_{\max} - p_{\min}}{1 + \exp(-x)}, \quad (75)$$

which guarantees $p_{\min} < p < p_{\max}$. The large-scale anisotropy levels are represented so that $\mu_{\ell,\text{max}} > 1$, ensuring that the remapping corresponds to an anisotropic stretching relative to the isotropic limit. The Bernstein coefficient matrices are also not



optimized directly. Instead, unconstrained variables are transformed through a cumulative softplus construction. Let $\vartheta_{ij}^{(\ell)} \in \mathbb{R}$ be unconstrained raw variables and define non-negative increments

$$d_{ij}^{(\ell)} = \text{softplus} \left(\vartheta_{ij}^{(\ell)} \right) = \log \left(1 + \exp \left(\vartheta_{ij}^{(\ell)} \right) \right) \geq 0, \quad (76)$$

with the corner increment fixed as $d_{00}^{(\ell)} = 0$. A cumulative surface is then formed as

$$Q_{ij}^{(\ell)} = \sum_{p=0}^i \sum_{q=0}^j d_{pq}^{(\ell)}. \quad (77)$$

The admissible Bernstein coefficients are obtained from

$$c_{ij}^{(\ell)} = 1 - \frac{Q_{ij}^{(\ell)}}{Q_{N_1, N_2}^{(\ell)}}. \quad (78)$$

This construction guarantees the conditions in Eqs. (16)–(17). Thus, every iterate of the optimizer corresponds to a physically admissible monotone anisotropy-decay surface. This is advantageous because the optimizer searches in an unconstrained Euclidean parameter space while the transformed parameters always satisfy the theoretical constraints of the model. The optimization itself is performed hierarchically using a staged multi-resolution strategy. In the first stage, the isotropic backbone is fitted with the anisotropic remapping disabled. This stage estimates the amplitude parameters and the isotropic spectral roll-off before directional spectral distortions are introduced. A variance-matching warm start is used for the amplitudes A_u and A_w , which improves the initial scaling of the model spectra and reduces the burden on the subsequent nonlinear optimization. In the second stage, the baseline backbone is frozen and the anisotropic and active geometric parameters are optimized. This includes the large-scale anisotropy levels $\mu_{1, \max}$ and $\mu_{2, \max}$, the raw Bernstein variables that generate $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$, and the four transition wavenumbers. The rotation angle remains fixed at $\theta = 0$. This stage identifies the bidirectional, scale-dependent deformation of the spectral coordinates while avoiding simultaneous compensation between the baseline spectral shape and the anisotropic remapping. In the third stage, all active parameters are jointly refined. This final refinement allows the isotropic backbone and the anisotropic correction to adjust consistently after the dominant anisotropic structure has already been identified.

The staged procedure therefore improves robustness relative to a single all-at-once optimization, which is more sensitive to local minima and parameter degeneracy. Each optimization stage is solved using the coarse-to-fine decimation sequence. The procedure begins on a block-averaged spectral grid, which captures the dominant large-scale spectral structure at reduced computational cost while approximately preserving the integrated spectral energy. The resulting parameter estimate is then used as the initialization for progressively finer grids, allowing the model to refine the anisotropy field and high-wavenumber behavior. This strategy improves numerical stability, reduces sensitivity to noise in the measured spectra, and helps avoid poor local minima. At each resolution level, the objective function is minimized using the L-BFGS-B algorithm. Although the physical constraints are already enforced by reparameterization, L-BFGS-B provides an efficient quasi-Newton update for the resulting nonlinear least-squares problem. If the L-BFGS-B step fails or produces an objective value worse than the starting point, a Nelder–Mead fallback is used as a derivative-free recovery step. After the staged and multi-resolution fitting is complete, the final parameter set is evaluated on the full spectral grid.



370 4 From fitted spectra to synthetic turbulence

The fitted planar spectra provide more than a compact description of the LES data in wavenumber space. They can also be used directly to generate synthetic velocity fields with prescribed spatial organization (Ali et al., 2024). Let $c \in \{u, w\}$ denote either the streamwise or wall-normal velocity fluctuation; throughout this section, the prime is suppressed in the generic symbol c . Let $E_c^M(k_1, k_2)$ be its fitted two-dimensional spectrum. Because the present model contains only the two component auto-spectra
 375 and no uw cross-spectrum, the u' and w' fields are synthesized independently; consequently, the synthetic Reynolds shear stress $\langle u'w' \rangle$ is zero by construction. Under the assumptions of statistical homogeneity and Gaussianity, this spectrum completely determines the second-order spatial statistics and, consequently, the joint probability law of the corresponding random field. This connection makes it possible to evaluate the spectral model in physical space. Synthetic realizations generated from E_c^M can be compared with the LES using statistics that were not included explicitly in the fitting objective, such as two-point
 380 correlations, integral length scales, and structure functions. These comparisons therefore provide an out-of-sample assessment of whether the fitted anisotropy reproduces the spatial dimensions and directional organization of the turbulent structures. A Gaussian spectral construction, however, cannot reproduce the skewness, heavy tails, or intermittency observed in the LES velocity distributions. To address this limitation, the Gaussian field is subsequently transformed using a monotone translation process. Two alternatives are considered, including a low-order Hermite transformation that matches prescribed skewness
 385 and kurtosis, and an empirical rank transformation that reproduces the complete one-point distribution. Because a nonlinear pointwise transformation modifies spatial correlations, the Gaussian spectrum is pre-distorted before the transformation is applied. This correction allows the final non-Gaussian field to retain the fitted spectrum to a controlled numerical accuracy.

4.1 Gaussian synthesis from the fitted spectrum

4.1.1 Unfolding the quadrant spectrum

390 The measured and modeled spectra use a one-sided, quadrant-folded convention. Under this convention, all distinct Fourier modes related by reflection are accumulated in the same positive-quadrant cell $(|k_1|, |k_2|)$. Synthetic field generation, by contrast, requires a spectral density over the complete discrete wavenumber plane. The fitted spectrum must therefore be unfolded before the Fourier coefficients are generated. Let $S_c(\mathbf{k})$, with $\mathbf{k} = (k_1, k_2)$, denote the full-plane spectral density, and define the set of distinct reflected modes

$$395 \mathcal{O}(k_1, k_2) = \{(s_1 k_1, s_2 k_2) : s_1, s_2 \in \{-1, 1\}\}.$$

The corresponding folded spectrum is

$$E_c^M(k_1, k_2) = \sum_{\mathbf{k}' \in \mathcal{O}(k_1, k_2)} S_c(\mathbf{k}'), \quad k_1, k_2 \geq 0, \quad (79)$$

where repeated points on the coordinate axes, the origin, and Nyquist lines are counted only once. For a real-valued field, the full-plane density satisfies the Hermitian symmetry implied by $S_c(-\mathbf{k}) = S_c(\mathbf{k})$. The present model is also reflection



400 symmetric with respect to the coordinate axes. Consequently, an interior positive-quadrant mode represents four distinct full-plane modes, whereas a mode on one coordinate axis represents only two. The origin, as well as a discrete Nyquist mode, is its own reflection image. Consider a spatial grid containing $N_1 \times N_2$ points with spacings Δx_1 and Δx_2 . The corresponding wavenumber increments and Nyquist wavenumbers are

$$\Delta k_i = \frac{2\pi}{N_i \Delta x_i}, \quad k_{\text{Nyq},i} = \frac{\pi}{\Delta x_i}, \quad i \in \{1, 2\}. \quad (80)$$

405 The full-plane density is then recovered from the fitted quadrant spectrum as

$$S_c(\mathbf{k}) = \frac{E_c^M(|k_1|, |k_2|)}{m_1(k_1) m_2(k_2)}, \quad m_i(k_i) = \begin{cases} 1, & k_i = 0 \text{ or } k_i = k_{\text{Nyq},i}, \\ 2, & \text{otherwise,} \end{cases} \quad (81)$$

where $S_c(\mathbf{0})$ is set to zero to enforce a zero-mean fluctuation field. Correct treatment of the axis and Nyquist modes is necessary for discrete energy conservation. With the multiplicities in Eq. (81), the full-plane Parseval sum reproduces the energy contained in the folded spectrum without introducing a boundary-counting error.

410 4.1.2 Spectral filtering of Gaussian white noise

We adopt the discrete Fourier-transform convention

$$\hat{c}(\mathbf{k}) = \sum_{\mathbf{x}} c(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad c(\mathbf{x}) = \frac{1}{N_1 N_2} \sum_{\mathbf{k}} \hat{c}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (82)$$

For this normalization, Parseval's identity gives

$$\overline{c^2} \equiv \frac{1}{N_1 N_2} \sum_{\mathbf{x}} c^2(\mathbf{x}) = \frac{1}{(N_1 N_2)^2} \sum_{\mathbf{k}} |\hat{c}(\mathbf{k})|^2. \quad (83)$$

415 Let $z(\mathbf{x})$ be a field of independent standard normal random variables, and let $\hat{Z}(\mathbf{k})$ denote its discrete Fourier transform. Its Fourier coefficients satisfy

$$\mathbb{E} [\hat{Z}(\mathbf{k}) \hat{Z}^*(\mathbf{k}')] = N_1 N_2 \delta_{\mathbf{k}\mathbf{k}'}. \quad (84)$$

A realization with the target spectral density $S_c(\mathbf{k})$ is obtained by filtering these coefficients according to

$$\hat{c}(\mathbf{k}) = \hat{Z}(\mathbf{k}) H_c(\mathbf{k}), \quad H_c(\mathbf{k}) = \sqrt{N_1 N_2 S_c(\mathbf{k}) \Delta k_1 \Delta k_2}, \quad c = \mathcal{F}^{-1} [\hat{c}]. \quad (85)$$

420 It follows that

$$\mathbb{E} [|\hat{c}(\mathbf{k})|^2] = (N_1 N_2)^2 S_c(\mathbf{k}) \Delta k_1 \Delta k_2. \quad (86)$$

Substitution into Eq. (83) yields

$$\mathbb{E} [\overline{c^2}] = \sum_{\mathbf{k}} S_c(\mathbf{k}) \Delta k_1 \Delta k_2, \quad (87)$$



showing that the ensemble variance of the synthetic field is equal to the discrete integral of the fitted spectrum. Because the
 425 filter is applied to a real Gaussian noise field, the Fourier coefficients inherit the required Hermitian symmetry. The inverse
 transform is therefore real to numerical precision, without discarding an independently generated imaginary component. The
 resulting field is also exactly Gaussian because it is obtained through a linear transformation of Gaussian random variables.
 For non-self-conjugate modes, the filtered Fourier coefficient is a circularly symmetric complex Gaussian variable. Its phase
 is uniformly distributed and its amplitude follows a Rayleigh distribution. The procedure is therefore equivalent to a spectral-
 430 representation method with random amplitudes and phases (Shinozuka and Deodatis, 1991). In contrast to fixed-amplitude
 spectral constructions, Gaussianity is retained for any finite number of Fourier modes rather than only in an asymptotic sense.
 The target spectrum is recovered exactly in the ensemble mean, but the periodogram of an individual realization fluctuates
 around that mean. For a Gaussian field, the modal energy $|\hat{c}(\mathbf{k})|^2$ is exponentially distributed and consequently has a relative
 standard deviation of 100% for an individual mode. This sampling variability decreases when spectra are averaged over neigh-
 435 boring modes, wavenumber rings, or independent realizations. Synthetic fields should therefore be processed using the same
 windowing, folding, and averaging operations applied to the LES. Because the underlying spectrum of the synthetic field is
 known, this comparison also helps distinguish spectral-estimator bias from deficiencies in the fitted model itself.

4.1.3 Physical-space validation

The spectral fitting procedure minimizes a weighted mismatch in wavenumber space. Correlations, integral scales, and structure
 440 functions are not fitted independently and can therefore be used to assess the physical-space implications of the model. Under
 the periodic and homogeneous assumptions used for synthesis, the normalized two-point correlation is obtained from the
 full-plane density through the discrete Wiener–Khinchin relation,

$$\rho_c(\mathbf{r}) = \frac{1}{c^2} \sum_{\mathbf{k}} S_c(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} \Delta k_1 \Delta k_2. \quad (88)$$

Directional integral length scales can then be defined as

$$445 \quad L_c^{(i)} = \int_0^{r_{0,i}} \rho_c(r \mathbf{e}_i) dr, \quad i \in \{1, 2\}, \quad (89)$$

where $r_{0,i}$ is the first zero crossing of the correlation in direction $\mathbf{e}_i$. The integration is restricted to separations smaller than half
 the domain length to prevent contamination by periodic wrap-around. For a homogeneous field, the directional second-order
 structure function is directly related to the correlation:

$$D_{2,c}^{(i)}(r) = \left\langle [c(\mathbf{x} + r \mathbf{e}_i) - c(\mathbf{x})]^2 \right\rangle = 2\overline{c^2} [1 - \rho_c(r \mathbf{e}_i)]. \quad (90)$$

450 Applying identical estimators to the LES and synthetic ensembles provides a consistent comparison at small and intermediate
 separations, even though the LES planes are not perfectly homogeneous or periodic. Agreement in the ratio $L_c^{(1)}/L_c^{(2)}$, together
 with agreement in the directional decay of ρ_c , indicates that the fitted anisotropy reproduces the physical elongation and
 orientation of the turbulent structures. Differences at the largest separations are expected to be more sensitive to finite-domain
 effects, spatial inhomogeneity, and the low-wavenumber axis modes excluded from the spectral fit.



4.2 Non-Gaussian synthesis by translation processes

The Gaussian construction reproduces the fitted second-order statistics but cannot represent non-Gaussian one-point behavior. LES velocity fluctuations may exhibit skewness, sharp central peaks, heavy tails, and intermittent extreme events. These characteristics can be introduced through a translation process (Grigoriu, 1998), in which an underlying Gaussian field is transformed pointwise as

$$c(\mathbf{x}) = \sigma_c h(g(\mathbf{x})), \quad g \sim \mathcal{N}(0, 1), \quad (91)$$

where the transformation h is selected such that

$$\mathbb{E}[h(g)] = 0, \quad \text{Var}[h(g)] = 1. \quad (92)$$

The map is required to be non-decreasing so that the ordering of the Gaussian samples is preserved and the transformation defines a valid correspondence between Gaussian and target quantiles. The function h controls the one-point probability distribution, whereas the correlation structure of g controls the spatial dependence. These two properties cannot be prescribed independently without correction. A nonlinear pointwise transformation changes the correlation function and therefore modifies the spectrum. The correlation distortion can be quantified explicitly using a Hermite expansion, which then provides a practical means of pre-distorting the Gaussian field before applying the transformation.

4.2.1 Correlation distortion and spectral pre-distortion

Expand the standardized transformation in the orthonormal probabilists' Hermite basis

$$\psi_n(g) = \frac{\text{He}_n(g)}{\sqrt{n!}}, \quad \mathbb{E}[\psi_n(g)\psi_m(g)] = \delta_{nm}, \quad (93)$$

so that

$$h(g) = \sum_{n=0}^{\infty} \alpha_n \psi_n(g), \quad \alpha_n = \mathbb{E}[h(g)\psi_n(g)]. \quad (94)$$

For two jointly Gaussian variables g and g' with correlation ρ_g , Mehler's identity gives

$$\mathbb{E}[\psi_n(g)\psi_m(g')] = \delta_{nm} \rho_g^n. \quad (95)$$

The correlation-transfer function associated with h is therefore

$$\mathcal{T}_h(\rho_g) = \sum_{n=1}^{\infty} \omega_n \rho_g^n, \quad \omega_n = \frac{\alpha_n^2}{\sum_{m=1}^{\infty} \alpha_m^2}, \quad \omega_n \geq 0, \quad \sum_{n=1}^{\infty} \omega_n = 1. \quad (96)$$

Several useful properties follow from this representation. First, $\mathcal{T}_h(1) = 1$, so perfect positive correlation is preserved. Second, the mapping is monotone on the positive-correlation interval and can therefore be inverted. Third,

$$\mathcal{T}_h(\rho_g) \leq \rho_g, \quad 0 \leq \rho_g \leq 1, \quad (97)$$



which shows that a nonlinear pointwise transformation generally weakens positive correlations. For negative correlations, the attainable lower limit is

$$\mathcal{T}_h(-1) = \sum_{n=1}^{\infty} \omega_n (-1)^n > -1. \quad (98)$$

A target correlation below this limit cannot be generated by the selected translation map. This is the classical correlation bound associated with translation processes. To preserve the fitted target spectrum, the correlation distortion is removed through spectral pre-distortion. Let $\rho_c^{\text{tar}}(\mathbf{r})$ be the target correlation obtained from the fitted spectrum using Eq. (88). At each spatial lag, the required Gaussian correlation is found by solving

$$\rho_c^{\text{tar}}(\mathbf{r}) = \sum_{n=1}^{n_{\max}} \omega_n [\rho_g(\mathbf{r})]^n. \quad (99)$$

The complete synthesis procedure begins by constructing the transformation h and computing its Hermite coefficients α_n up to the prescribed order $n_{\max}$, while recording the fraction of transformation variance omitted by the truncation. The target correlation field, $\rho_c^{\text{tar}}(\mathbf{r})$, is then obtained from the unfolded fitted spectrum through an inverse Fourier transform. Eq. (96) is subsequently inverted independently at each spatial lag to determine the corresponding Gaussian correlation $\rho_g(\mathbf{r})$. For the cubic Hermite transformation, this inversion is performed using Newton iteration, whereas bisection is employed for a general empirical transformation. Any target correlations falling below the attainable limit defined by Eq. (98) are clipped to that bound. The resulting Gaussian correlation field is then converted into a Gaussian spectral density using a forward Fourier transform. Small negative spectral values arising from numerical discretization or correlation inversion are set to zero, and the associated removed energy fraction, denoted by $\varepsilon_{\text{clip}}$, is recorded. The remaining spectrum is subsequently renormalized to unit variance. Finally, the Gaussian substrate $g(\mathbf{x})$ is generated according to Eq. (85), after which the pointwise transformation in Eq. (91) is applied. The resulting field is then rescaled using the variance obtained by integrating the fitted model spectrum. The accuracy of the resulting second-order statistics is controlled by the Hermite-series truncation, the negative-spectrum clipping fraction $\varepsilon_{\text{clip}}$, and the attainable negative-correlation bound. In the present calculations, $\varepsilon_{\text{clip}}$ is of order 10^{-4} , while truncation at $n_{\max} = 40$ leaves less than 10^{-5} of the transformation variance unresolved. Moreover, the lower correlation bound is approximately $\mathcal{T}_h(-1) \lesssim -0.94$, which is below the minimum correlations encountered in the fitted fields. These errors are therefore small compared with the uncertainty associated with finite-sample spectral estimation.

4.2.2 Moment-matched cubic Hermite transformation

The Gaussian spectral synthesis described above reproduces the prescribed second-order spatial statistics, but it cannot represent the asymmetric or heavy-tailed one-point distributions that may arise in atmospheric-boundary-layer and wind-turbine-wake turbulence. To introduce controlled non-Gaussianity, the standardized Gaussian field is passed through a pointwise cubic Hermite transformation. Moment-based Hermite transformations have been used extensively to construct non-Gaussian stochastic processes from an underlying Gaussian process while retaining a compact parametric description (Grigoriu, 1995). Let $g(\mathbf{x})$ denote a zero-mean, unit-variance Gaussian random field. The standardized non-Gaussian field is defined as

$$h(g) = \Lambda [g + h_3 (g^2 - 1) + h_4 (g^3 - 3g)], \quad (100)$$



where

$$\Lambda = (1 + 2h_3^2 + 6h_4^2)^{-1/2}. \quad (101)$$

515 The quantities $\text{He}_1 = g$, $\text{He}_2 = g^2 - 1$, $\text{He}_3 = g^3 - 3g$ are the first three non-constant probabilists' Hermite polynomials. Their orthogonality under the standard Gaussian measure gives

$$\mathbb{E}[\text{He}_m(g)\text{He}_n(g)] = n! \delta_{mn}, \quad (102)$$

and therefore

$$\text{Var}[h(g)] = \Lambda^2 (1 + 2h_3^2 + 6h_4^2) = 1. \quad (103)$$

520 Thus, the transformation changes the shape of the marginal distribution without altering its prescribed variance. The dimensional synthetic fluctuation is subsequently obtained as

$$c_{\text{NG}}(\mathbf{x}) = \sigma_c h(g(\mathbf{x})), \quad (104)$$

where σ_c^2 is the variance obtained by integrating the corresponding fitted two-dimensional spectrum. The coefficient h_3 primarily controls distributional asymmetry, whereas h_4 controls the departure of the tails and central peak from the Gaussian form. Their effects are not completely independent, however, because both parameters contribute to the third and fourth standardized moments of the transformed variable. Rather than relying directly on approximate moment relations, the present implementation determines h_3 and h_4 by solving

$$\begin{aligned} \mathcal{S}(h_3, h_4) &= \frac{\mathbb{E}[h(g)^3]}{\mathbb{E}[h(g)^2]^{3/2}} = \mathcal{S}_{\text{tar}}, \\ \mathcal{K}(h_3, h_4) &= \frac{\mathbb{E}[h(g)^4]}{\mathbb{E}[h(g)^2]^2} = \mathcal{K}_{\text{tar}}, \end{aligned} \quad (105)$$

where $\mathcal{S}_{\text{tar}}$ and $\mathcal{K}_{\text{tar}}$ are the skewness and kurtosis measured from the LES data. The classical moment-based expressions of Wint'ersstein (1988) are used only to provide an initial estimate. The expectations in Eq. (105) are then evaluated by an 80-point Gauss–Hermite quadrature, which is effectively exact for the polynomial moments involved here, and the resulting nonlinear system is solved by a damped Newton iteration. For a translation process, the mapping should be non-decreasing so that the ordering of the underlying Gaussian samples is preserved. Differentiating Eq. (100) gives

$$h'(g) = \Lambda [3h_4 g^2 + 2h_3 g + (1 - 3h_4)]. \quad (106)$$

535 Since the expression inside the brackets is quadratic in g , global monotonicity is obtained when

$$0 \leq 3h_4 < 1, \quad h_3^2 \leq 3h_4(1 - 3h_4). \quad (107)$$

These inequalities define the feasible parameter region of the monotone cubic model. During the Newton iteration, each update is projected back into this region. Consequently, when a prescribed skewness–kurtosis pair cannot be represented exactly by a



monotone cubic transformation, the algorithm returns the closest attainable solution encountered at the feasibility boundary and reports both the requested and achieved moments. This limitation is particularly relevant for strongly platykurtic distributions and for distributions whose skewness is too large relative to their kurtosis. A nonlinear pointwise transformation generally changes not only the marginal distribution but also the two-point correlation. Let g_1 and g_2 be jointly Gaussian standard variables with $\rho_g = \mathbb{E}[g_1 g_2]$. Hermite orthogonality gives the exact transformed correlation

$$\rho_h = \Lambda^2 (\rho_g + 2h_3^2 \rho_g^2 + 6h_4^2 \rho_g^3). \quad (108)$$

The relation terminates at third order because the transformation contains only the first three non-constant Hermite modes. Directly applying Eq. (100) to a Gaussian field having the target spectrum would therefore distort the prescribed second-order spatial structure. To compensate for this effect, the Gaussian correlation is pre-distorted before the random field is synthesized. For a desired target correlation $\rho_c^{\text{tar}}(\mathbf{r})$, the required Gaussian correlation is obtained from

$$\rho_g(\mathbf{r}) = \mathcal{T}_H^{-1} [\rho_c^{\text{tar}}(\mathbf{r})], \quad (109)$$

where

$$\mathcal{T}_H(\rho) = \Lambda^2 (\rho + 2h_3^2 \rho^2 + 6h_4^2 \rho^3). \quad (110)$$

In the implementation, this inverse is evaluated pointwise by a bounded Newton iteration on $[-1, 1]$. The pre-distorted Gaussian spectral density is then computed by Fourier transformation of ρ_g . Small negative values may appear because the inverse correlation is not guaranteed to be positive definite on a finite discrete grid. These values are set to zero, and the remaining spectrum is renormalized to unit Gaussian variance. The fraction of negative spectral mass removed in this step is reported as a measure of the second-order inconsistency introduced by the projection. The moment-matched Hermite representation is therefore attractive when only skewness and kurtosis are available or when a compact two-parameter model is required. It provides transparent control of the leading non-Gaussian moments and admits an analytical correlation-transformation relation. Its principal limitation is that agreement in the third and fourth moments does not imply agreement with the complete marginal distribution. In particular, multimodal behavior, sharp central peaks, intermittent mixtures, and complex asymmetric tails may remain unresolved even when the prescribed skewness and kurtosis are matched accurately.

4.2.3 Empirical quantile transformation

The full sampled one-point distribution can be retained without imposing a low-order parametric form. This is achieved through a translation-process or quantile transformation, in which a Gaussian variable is mapped onto the empirical LES distribution (Grigoriu, 1998; Yamazaki and Shinozuka, 1988). Such transformations provide a general framework for simulating non-Gaussian stochastic fields with prescribed marginal and correlation properties. Let $F_{\mathcal{N}}$ denote the standard-normal cumulative distribution function and let $\hat{F}_{\text{LES}}$ denote the empirical cumulative distribution function of the standardized LES fluctuations. The transformation is

$$h(g) = \hat{F}_{\text{LES}}^{-1} [F_{\mathcal{N}}(g)]. \quad (111)$$



570 Since $F_{\mathcal{N}}(g)$ is uniformly distributed on $(0, 1)$, the probability integral transform gives

$$\begin{aligned}\mathbb{P}[h(g) \leq x] &= \mathbb{P}\left[\widehat{F}_{\text{LES}}^{-1}(F_{\mathcal{N}}(g)) \leq x\right] \\ &= \mathbb{P}\left[F_{\mathcal{N}}(g) \leq \widehat{F}_{\text{LES}}(x)\right] \\ &= \widehat{F}_{\text{LES}}(x).\end{aligned}\tag{112}$$

The transformed variable therefore follows the empirical LES marginal distribution, apart from interpolation and finite-sample effects. In the present implementation, the LES fluctuations are first centered and standardized using their sample mean and standard deviation. Because the full space–time LES archive may contain substantially more values than are required to resolve the marginal distribution, the empirical map is constructed from a random subsample of approximately $N_{\text{sam}} \simeq 2 \times 10^6$. The data are processed in temporal chunks, and each value is retained with a sampling probability selected to achieve the requested sample size. This procedure limits the additional memory requirement while preserving samples from the complete LES record rather than from a small set of selected snapshots. After sorting the standardized samples, $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(N)}$ the empirical probability assigned to $x_{(j)}$ is

$$580 \quad p_j = \frac{j - \frac{1}{2}}{N}, \quad j = 1, \dots, N.\tag{113}$$

The inverse empirical distribution in Eq. (111) is then represented by piecewise-linear interpolation between the pairs $(p_j, x_{(j)})$. The normal probability is restricted to

$$F_{\mathcal{N}}(g) \in \left[\frac{1}{2N}, 1 - \frac{1}{2N}\right].\tag{114}$$

Consequently, values outside the empirically resolved probability range are mapped to the smallest or largest sampled LES fluctuation. This bounded extension avoids unsupported extrapolation beyond the observed data range. It should nevertheless be recognized that the synthetic extremes are then limited by the available LES record. When rare-event extrapolation is required, the central empirical distribution could instead be joined continuously to a parametric extreme-value tail model. As with the cubic transformation, applying the empirical map directly to a Gaussian field modifies its correlation and spectrum. Since the empirical quantile map has no simple finite-order analytical correlation relation, it is expanded in the same orthonormal probabilists' Hermite basis ψ_n introduced above,

$$590 \quad h(g) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \psi_n(g),\tag{115}$$

with

$$\alpha_n = \mathbb{E}[h(g)\psi_n(g)].\tag{116}$$

For numerical evaluation, the orthonormal polynomials are generated recursively as

$$595 \quad \psi_0(g) = 1, \quad \psi_1(g) = g,\tag{117}$$



and

$$\psi_{n+1}(g) = \frac{g\psi_n(g) - \sqrt{n}\psi_{n-1}(g)}{\sqrt{n+1}}. \quad (118)$$

For two standard Gaussian variables with correlation ρ_g , Mehler's identity gives

$$\mathbb{E}[\psi_n(g_1)\psi_m(g_2)] = \rho_g^n \delta_{mn}. \quad (119)$$

600 It follows that the covariance of the transformed variables is

$$\text{Cov}[h(g_1), h(g_2)] = \sum_{n=1}^{\infty} \alpha_n^2 \rho_g^n. \quad (120)$$

After normalization by the transformed variance,

$$\sigma_h^2 = \sum_{n=1}^{\infty} \alpha_n^2, \quad (121)$$

the correlation relation becomes

$$605 \quad \rho_h = \mathcal{T}_R(\rho_g) = \sum_{n=1}^{\infty} \omega_n \rho_g^n, \quad \omega_n = \frac{\alpha_n^2}{\sum_{m=1}^{\infty} \alpha_m^2}. \quad (122)$$

The coefficients satisfy $\omega_n \geq 0$ and $\sum_{n=1}^{\infty} \omega_n = 1$. Eq. (122) generalizes the finite cubic relation in Eq. (108) to an arbitrary monotone marginal transformation. The numerical implementation truncates the expansion at

$$n_{\max} = 40. \quad (123)$$

The Hermite coefficients are evaluated using a Gaussian-weighted trapezoidal quadrature on

$$610 \quad -9 \leq g \leq 9. \quad (124)$$

A dense trapezoidal rule is used instead of very-high-order Gauss–Hermite quadrature because the latter can become poorly conditioned when evaluating many high-order polynomial terms with extremely small quadrature weights. The bounded empirical transformation and the rapidly decaying Gaussian density make the finite interval sufficient for the expansion orders considered here. The unresolved fraction of the transformed variance is monitored through

$$615 \quad \varepsilon_H = \max \left[0, 1 - \frac{\sum_{n=1}^{n_{\max}} \alpha_n^2}{\text{Var}[h(g)]} \right]. \quad (125)$$

This quantity provides a direct diagnostic of whether the selected expansion order adequately represents the empirical map. A small value of ε_H indicates that nearly all marginal variance is contained in the retained Hermite modes. For each spatial lag, the target correlation is converted to the corresponding Gaussian correlation by solving

$$\mathcal{T}_R(\rho_g) = \rho_c^{\text{tar}}. \quad (126)$$



620 The truncated map is checked numerically for monotonicity over $[-1, 1]$, and Eq. (126) is inverted by bisection. This approach is slower than the Newton inversion used for the cubic map but is robust and requires no analytical derivative. When a target negative correlation lies below the minimum attainable value $\mathcal{T}_R(-1)$, it is clipped to that limit, reflecting a known restriction of translation-process models. The pre-distorted Gaussian correlation is subsequently Fourier transformed to obtain the Gaussian spectral density used by the random-field generator. As in the cubic formulation, small negative spectral values may arise if

625 the pre-distorted correlation is not positive definite on the discrete periodic grid. These values are removed, and the spectral density is renormalized. The discarded negative spectral fraction,

$$\varepsilon_{\text{clip}} = \frac{-\sum_{\mathbf{k}: S_g(\mathbf{k}) < 0} S_g(\mathbf{k})}{\sum_{\mathbf{k}} |S_g(\mathbf{k})|}, \quad (127)$$

is reported together with ε_H . The former measures the incompatibility between the target correlation, the selected marginal distribution, and the finite spectral grid, whereas the latter measures only the truncation of the Hermite representation. Finally,

630 the transformed field is centered and scaled according to

$$c_{\text{rank}}(\mathbf{x}) = \left(\frac{\sigma_c^2}{\sigma_h^2} \right)^{1/2} [h(g(\mathbf{x})) - \alpha_0], \quad (128)$$

where σ_c^2 is obtained by integrating the fitted two-dimensional spectrum. The final field therefore combines the empirical marginal shape with the variance and scale-dependent directional structure of the fitted spectral model. The Hermite and rank transformations serve complementary purposes. The Hermite method gives a compact moment-based representation and is

635 appropriate when skewness and kurtosis are the primary non-Gaussian quantities of interest. The rank method uses more information from the LES and reproduces the complete sampled marginal distribution, including its central peak and resolved tails. In both cases, spectral pre-distortion separates the specification of the one-point distribution from the specification of the second-order spatial structure. The resulting synthetic fields can therefore retain the scale-dependent anisotropy encoded by the fitted two-dimensional spectrum while also reproducing the non-Gaussian amplitude statistics of the underlying turbulent

640 flow.

5 Simulations and test cases

We evaluate the method using open-access large-eddy simulation (LES) datasets that provide two single-wake cases under low and high ambient turbulence intensities (Van der Laan and Andersen, 2018). The reference turbine is the NREL 5-MW machine. The rotor is modeled with an actuator-disk (AD) using a time-independent body-force distribution and a constant

645 thrust coefficient $C_T = 0.79$ in a neutral atmospheric surface layer. The hub-height wind speed is established as $U_h = 8 \text{ ms}^{-1}$, the rotor diameter is $D = 126 \text{ m}$, and the hub height is $z_h = 90 \text{ m}$. The flow is solved with ELLIPSYS3D, which supports both Reynolds-Averaged Navier-Stokes (RANS) and LES (Michelsen, 1994). Ambient turbulence is generated with the Mann model and imposed on a plane $1.5D$ upstream of the AD. A slip-wall boundary condition allows the inflow profile and Mann



turbulence to evolve downstream; the adjustments remain small, so the wake is representative of a near-equilibrium turbulent
650 shear flow. Coordinates (x, y, z) denote the streamwise, lateral, and wall-normal directions. LES fields are sampled on the hub-
height plane over $x \in [-1D, 8.5D]$ and $y \in [-2D, 2D]$. The AD is located at $(x, y, z) = (0, 0, 90 \text{ m})$. Each case covers one hour
of simulation, which is sufficient for converged low-order statistics. Two wind-turbine wake-flow cases are considered, cor-
responding to low and high turbulence-intensity conditions. The low-turbulence-intensity (LTI) and high-turbulence-intensity
(HTI) cases have incoming turbulence intensities of 4% and 12.8%, respectively. Both cases use the same spatial and temporal
655 discretization. The flow field is resolved using $N_x = 286$ grid points in the streamwise direction and $N_y = 121$ grid points in
the spanwise direction, with spatial resolutions of $\Delta_x = \Delta_y = 4.2 \text{ m}$. For each case, a total of $N_t = 15,500$ temporal snap-
shots are available, sampled at a temporal resolution of $\Delta_t = 0.24 \text{ s}$. In addition, an atmospheric boundary-layer (ABL) case
is included as a reference to support and contextualize the framework developed in the present study. The ABL case has the
same turbulence intensity, spatial and temporal resolutions, grid dimensions, and number of snapshots as the HTI case. For full
660 details on the simulations and data, the reader is referred to Van der Laan and Andersen (2018).

6 Results

The results are organized to assess the proposed framework from four complementary perspectives. First, the fitted parameters
are presented based on different stages of optimization. Second, the ability of the model to reconstruct the measured planar
spectra is examined for the ABL, WT-HTI, and WT-LTI cases. Third, the fitted anisotropy fields are then analyzed to determine
665 how the strength and directional dependence of the spectral deformation vary across scales and operating conditions. Finally,
the non-Gaussian synthesis results are evaluated to establish whether the fitted second-order spectral structure can be combined
with realistic one-point velocity statistics. Together, these comparisons provide a systematic assessment of the model in terms
of spectral accuracy, physical interpretability, and its suitability for generating statistically representative turbulent flow fields.

6.1 Fitted bidirectional anisotropic spectrum model

670 This section presents the fitted bidirectional anisotropic spectral model for three flow configurations. Figure 1 summarizes
the complete workflow of the proposed bidirectional, scale-dependent anisotropic spectral modeling framework, from the
estimation of planar component spectra to constrained parameter optimization, physical interpretation, and validation. The
complete set of fitted parameters is reported in Tables 1–4, with each table comparing the three cases side by side. Table 1 lists
the isotropic backbone and dissipation-correction parameters. The WT-LTI case exhibits by far the largest energy-containing
675 wavenumber k_e and the steepest roll-off exponent $7/3 + \beta$, whereas the ABL and WT-HTI cases share nearly identical values
of k_e , consistent with larger energy-containing scales in these flows. The LTI case also carries the largest backbone amplitudes
 A_u and A_w together with the smallest streamwise variance, so its spectrum is both weaker in total energy and concentrated
at smaller scales. Table 2 reports the large-scale anisotropy amplitudes and the logarithmic transition-scale factors entering
the Jacobian. In all three cases, $\mu_{2,\max} - 1 \simeq 1 \times 10^{-6}$, i.e. the fitted second-direction deformation is at its lower bound;

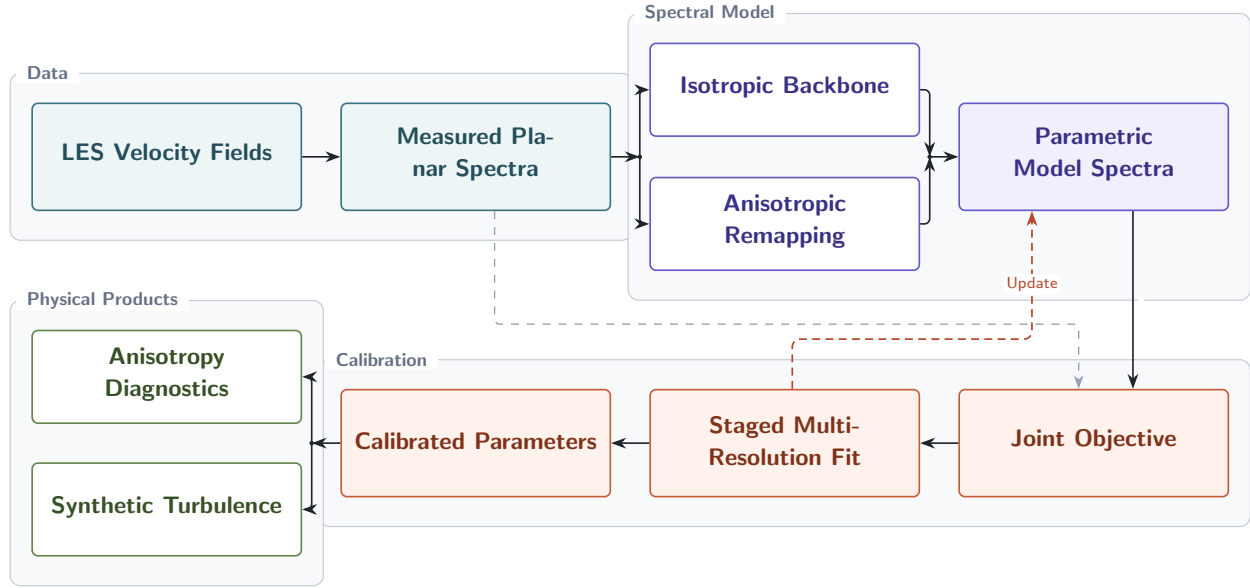


Figure 1. Structure of the proposed bidirectional, scale-dependent anisotropic spectral modeling framework.

Table 1. Fitted isotropic-backbone and dissipation-correction parameters, together with the fixed component variances used for normalization, for the three cases.

Parameter	ABL	WT-HTI	WT-LTI
A_u	5.486821	3.874298	9.324611
$\overline{u'^2}$	1.715212	2.478961	0.885407
A_w	8.557059	5.397274	11.956553
$\overline{w'^2}$	0.442797	0.980732	0.494869
k_e	0.021467	0.020496	0.052611
β	0.260088	0.048742	0.398589
η	2.354735	2.124564	2.054015
ξ	6.579217	4.835759	6.825476
c_η	1.112203	0.892787	1.314424

the physically meaningful anisotropy is therefore carried almost entirely by μ_1 and Ψ_1 . The weights used in all runs are $\lambda_u = \lambda_w = \lambda_E = \lambda_J = 1$ and $\lambda_S = 0.01$.

The transition scales in Table 2 are remarkably consistent across the three cases, where the first-direction band spans $k_{f,1} = 0.01 \text{ rad m}^{-1}$ to $k_{s,1} \approx 0.57\text{--}0.65 \text{ rad m}^{-1}$ ($\mathcal{R}_1 \approx 4.0\text{--}4.2$), and the second-direction band is identical in all three fits ($k_{f,2} = 0.1 \text{ rad m}^{-1}$, $k_{s,2} = 0.65 \text{ rad m}^{-1}$, $\mathcal{R}_2 = 1.8718$). Because $k_{f,1}$, $k_{f,2}$, and $k_{s,2}$ coincide with the edges of the admissible search



Table 2. Fitted large-scale anisotropy amplitudes, transition wavenumbers, and logarithmic transition-scale factors $\mathcal{R}_i = \ln(k_{s,i}/k_{f,i})$. Wavenumbers are in rad m^{-1} .

Parameter	ABL	WT-HTI	WT-LTI
$\mu_{1,\max} - 1$	0.559531	0.421193	0.496102
$\mu_{2,\max} - 1$	1×10^{-6}	1×10^{-6}	1×10^{-6}
$k_{f,1}$	0.010000	0.010000	0.010000
$k_{s,1}$	0.569028	0.649960	0.618619
$k_{f,2}$	0.100000	0.100000	0.100000
$k_{s,2}$	0.650000	0.650000	0.650000
$\mathcal{R}_1$	4.041344	4.174325	4.124904
$\mathcal{R}_2$	1.871802	1.871802	1.871802

Table 3. Power-basis coefficients $p_{mn}^{(1)}$ of the first anisotropy-decay surface $\Psi_1(\hat{q}_1, \hat{q}_2)$. The constant term is $p_{00}^{(1)} = 1$ in all cases; entries equal to zero correspond to monomials absent from the fitted expansion.

Monomial	ABL	WT-HTI	WT-LTI
$\hat{q}_2$	-3.2×10^{-6}	-0.383322	-0.391238
$\hat{q}_2^2$	-0.999988	-0.616677	-0.608761
$\hat{q}_1$	-2.3×10^{-13}	0	-2.5×10^{-13}
$\hat{q}_1 \hat{q}_2$	-1.6×10^{-9}	-1.1×10^{-9}	-1.9×10^{-10}
$\hat{q}_1 \hat{q}_2^2$	-1.8×10^{-5}	-1.7×10^{-6}	-2.6×10^{-6}
$\hat{q}_1^2$	$+1.2 \times 10^{-13}$	0	$+1.2 \times 10^{-13}$
$\hat{q}_1^2 \hat{q}_2$	$+7.8 \times 10^{-10}$	$+5.6 \times 10^{-10}$	$+9.5 \times 10^{-11}$
$\hat{q}_1^2 \hat{q}_2^2$	$+9.2 \times 10^{-6}$	$+8.2 \times 10^{-7}$	$+1.3 \times 10^{-6}$

interval in every case, these three quantities should be read as bound-limited rather than as independently resolved features of the data; only $k_{s,1}$ is fitted in the interior, and it is largest for the WT-HTI case, indicating that the deformation persists to slightly smaller scales there.

Tables 3 and 4 report the power-basis coefficients $p_{mn}^{(\ell)}$ of the two anisotropy-decay surfaces. In all three cases Ψ_1 is, to within a few parts in 10^5 , a function of $\hat{q}_2$ alone, where every monomial containing $\hat{q}_1$ carries a coefficient of order 10^{-5} or smaller, so the relaxation of the dominant anisotropy is controlled entirely by the second spectral coordinate. The coefficients of each surface sum with the constant term to zero, so that $\Psi_\ell \rightarrow 0$ at the upper corner of the normalized band and the mapping returns smoothly to the isotropic backbone beyond $k_{s,i}$. The three cases differ in how that relaxation is distributed between the linear and quadratic terms. For the ABL the decay is almost purely quadratic ($p_{02}^{(1)} = -1.000$ with a negligible linear coefficient), so the large-scale anisotropy is retained over most of the transition band and then collapses rapidly near $k_{s,2}$.



Table 4. Power-basis coefficients $p_{mn}^{(2)}$ of the second anisotropy-decay surface $\Psi_2(\hat{q}_1, \hat{q}_2)$. The constant term is $p_{00}^{(2)} = 1$ in all cases.

Monomial	ABL	WT-HTI	WT-LTI
$\hat{q}_2$	-0.898611	-0.689284	-0.757750
$\hat{q}_2^2$	-0.000444	+0.001376	-0.003814
$\hat{q}_1$	-0.098624	-0.316602	-0.235479
$\hat{q}_1 \hat{q}_2$	-1.4×10^{-5}	-3.1×10^{-6}	-7.2×10^{-6}
$\hat{q}_1 \hat{q}_2^2$	-0.001311	-2.8×10^{-5}	-0.000145
$\hat{q}_1^2$	+0.000744	+0.004543	-0.002373
$\hat{q}_1^2 \hat{q}_2$	-0.002730	-1.4×10^{-5}	-0.000624
$\hat{q}_1^2 \hat{q}_2^2$	+0.000989	$+1.2 \times 10^{-5}$	+0.000192

695 The two wake cases share a nearly identical mixed profile (roughly $-0.39\hat{q}_2 - 0.61\hat{q}_2^2$), giving an earlier and more gradual isotropization. The Ψ_2 surfaces retain genuine dependence on both coordinates, with a dominant $\hat{q}_2$ term and a secondary linear $\hat{q}_1$ term that is strongest for WT-HTI (-0.317) and weakest for the ABL (-0.099). Their contribution to the physical mapping is nevertheless multiplied by $\mu_{2,\max} - 1 \simeq 10^{-6}$, which sits at its lower bound in all three fits, so the detailed shape of Ψ_2 is weakly influential and effectively unidentifiable; it is reported for completeness rather than for physical interpretation.

700 Comparing the three columns of Tables 1–4 directly reveals the physical trends discussed in Sect. 6. Within the two wake cases, the LTI wake retains the stronger large-scale anisotropy amplitude ($\mu_{1,\max} - 1 = 0.496$) while the HTI wake exhibits the weaker deformation ($\mu_{1,\max} - 1 = 0.421$), consistent with enhanced ambient mixing under high inflow turbulence. The ABL reference carries the largest amplitude overall ($\mu_{1,\max} - 1 = 0.560$), although its anisotropy is associated primarily with boundary-layer shear rather than turbine-generated wake dynamics, and it is distinguished from the wake cases less by its

705 magnitude than by its purely quadratic decay profile, which keeps the deformation active across the full energy-containing range.

Two structural properties of the objective explain why the second stretch amplitude settles at its lower bound. The first concerns the energy constraint. Because the cumulative-softplus construction keeps each decay function between zero and one, the stretch factors are bounded between unity and their large-scale maxima, and the remapping is consequently a smooth,

710 orientation-preserving, one-to-one transformation of the positive wavenumber quadrant onto itself. Any such transformation conserves the integral of the spectral density, because the Jacobian factor multiplying the remapped baseline is exactly the volume element demanded by the change of variables. The component-variance term is therefore insensitive to the deformation by construction, where it fixes the backbone amplitudes and nothing more, apart from a negligible contribution from the spectral flux across the outer edge of the fitted band. The second property concerns the identifiability of the anisotropy. If the stretch

715 factors are independent of scale, their effect on the azimuthally averaged spectrum would reduce to a constant multiplicative factor and would therefore be indistinguishable from a rescaling of the backbone amplitude. Scale-independent anisotropy is thus exactly degenerate with amplitude in any radially averaged measurement. This is the spectral analogue of the cancellation

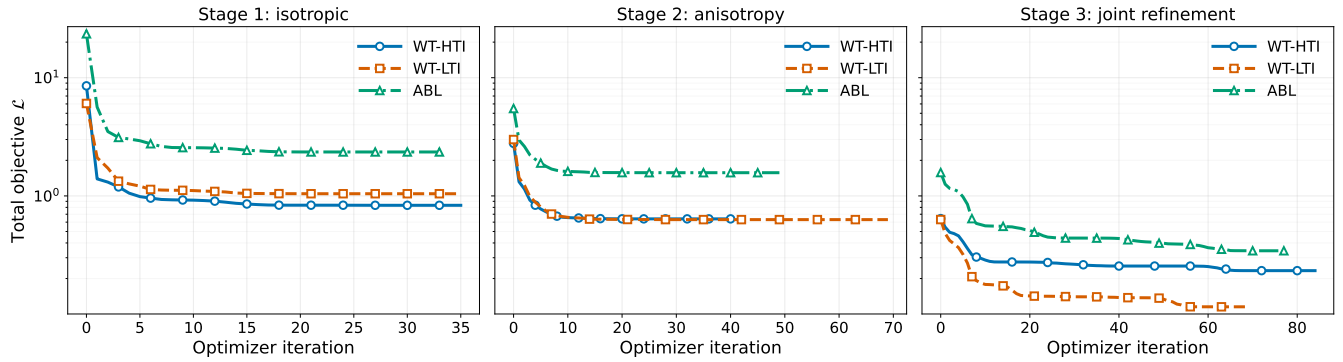


Figure 2. Training-loss for the WT-HTI, WT-LTI, and ABL cases. The curves show the total objective, $\mathcal{L}$, as a function of the L-BFGS-B optimizer iteration.

described by Zhai (2020), who showed that an anisotropy factor calibrated using a scalar propagation observable cancels from the resulting prediction. In the present formulation, both degeneracies are broken. First, the objective is evaluated over the complete planar spectrum rather than its radial projection, thereby retaining the directional information lost through azimuthal averaging. Second, when the stretch factors vary with scale, their angular signature also changes with wavenumber. This scale-dependent deformation modifies the curvature of the radial spectrum in a manner that cannot be reproduced by adjusting the component amplitudes alone.

Fig. 2 compares the convergence of the total objective, ($\mathcal{L}$), for the WT-HTI, WT-LTI, and ABL cases across the three-stage optimization procedure. In each stage, the objective decreases rapidly during the initial L-BFGS-B iterations and subsequently approaches a plateau, indicating stable numerical convergence. The different curve lengths arise from case-dependent termination once the optimizer satisfies its convergence criterion. During the isotropic stage, WT-HTI reaches the lowest objective, followed by WT-LTI, whereas the ABL case converges to a substantially higher value. This result indicates that the isotropic backbone alone provides a better representation of the two wake-flow spectra than of the ABL spectrum, although appreciable unresolved structure remains in all three cases. Introducing the scale-dependent anisotropic correction in stage 2 produces a further reduction in the objective. The WT-HTI and WT-LTI cases converge to nearly identical values, while the ABL objective remains higher, consistent with a more complex directional and scale-dependent spectral organization. The strongest additional improvement occurs during the joint-refinement stage, where the isotropic and anisotropic parameter blocks are optimized simultaneously. All three objectives decrease further, with WT-LTI attaining the lowest final value, followed by WT-HTI and ABL. The continued reduction during this stage demonstrates that the isotropic backbone and anisotropic remapping are coupled and benefit from simultaneous adjustment rather than being optimized independently.

6.2 Reconstruction of the planar velocity spectra

Figs. 3, 4, and 5 compare the planar wavenumber spectra obtained from the LES with the corresponding fitted anisotropic spectra for the ABL reference, WT-HTI, and WT-LTI cases, respectively. Each figure shows, for the streamwise and the wall-



740 normal component, the LES planar spectrum and the modeled planar spectrum in the (k_y, k_x) plane, colored by the base-ten logarithm of the spectral density, together with the corresponding radially averaged spectra. In all three configurations, the planar spectral energy is concentrated near the origin and decreases continuously toward the small scales. The decay is faster in the streamwise wavenumber k_x than in the spanwise wavenumber k_y , so that the isolines of the spectral density are elongated along the k_y axis, consistent with streamwise-elongated structures. The model reproduces both this broadband decay and
745 the directional spreading of the spectral density in the (k_y, k_x) plane, and the modeled and LES contour patterns remain comparable over the full plotted dynamic range. The similarity between the LES and modeled contour patterns is important because the model was fitted to the full planar spectrum rather than only to its radial projection. The radially averaged spectra provide a complementary comparison of the overall distribution of energy across scales, where $k_r = \sqrt{k_1^2 + k_2^2}$ denotes the physical radial wavenumber. For both the streamwise and wall-normal velocity components, the modeled spectra follow the
750 LES spectral envelope over the entire resolved wavenumber interval, $k_r \approx 5 \times 10^{-3} - 1 \text{ rad m}^{-1}$, and across six to seven orders of magnitude in spectral density.

In particular, the model captures the transition from the energy-containing range to the steep high-wavenumber attenuation. This behavior demonstrates that the anisotropic coordinate transformation and the dissipative correction can be combined without introducing an artificial discontinuity between the large- and small-scale portions of the spectrum. The main differences
755 occur at the lowest resolved wavenumbers, where the LES radial spectra contain a sequence of narrow peaks and troughs that is not reproduced by the smooth parametric model. These fluctuations are especially visible in the streamwise spectra and are associated with the small number of Fourier modes available at the largest scales, together with coherent and spatially inhomogeneous motions in the ABL and wake fields. A second, much weaker difference is visible in the wall-normal spectra, where the modeled curve falls slightly below the LES envelope over the last part of the roll-off before both collapse at the
760 highest resolved bins. The model represents the average spectral envelope and is not intended to reproduce individual low-wavenumber peaks. The use of ring-based spectral weighting also prevents a small number of energetic modes from dominating the complete planar fit.

For the ABL case in Fig. 3, the modeled streamwise spectrum closely reproduces the broad low-wavenumber energy concentration and the subsequent spectral roll-off. The model smooths the local oscillations of the LES radial spectrum but follows
765 its mean level over the intermediate and high wavenumbers. A similarly close agreement is obtained for the wall-normal component, although the modeled curve lies slightly above the LES spectrum in the first radial bins, $k_r \lesssim 2 \times 10^{-2} \text{ rad m}^{-1}$, and slightly below it during the final part of the roll-off. The agreement for both components shows that a common anisotropic mapping can describe the dominant directional organization of the ABL spectra while allowing the two reference component spectra to retain different spectral shapes; the wall-normal planar spectrum is visibly less elongated than the streamwise one,
770 and this difference is recovered by the fit. The WT-HTI results in Fig. 4 show similarly good agreement. The model reproduces the shape and extent of the energetic region in both component spectra and closely follows the LES decay at intermediate and high wavenumbers, remaining marginally above the LES streamwise spectrum over the roll-off. In the streamwise component, the LES spectrum contains an isolated peak near $k_r \approx 10^{-2} \text{ rad m}^{-1}$ that rises about one order of magnitude above the local envelope and is not reproduced by the smooth model. The strongest remaining discrepancy is again located in the

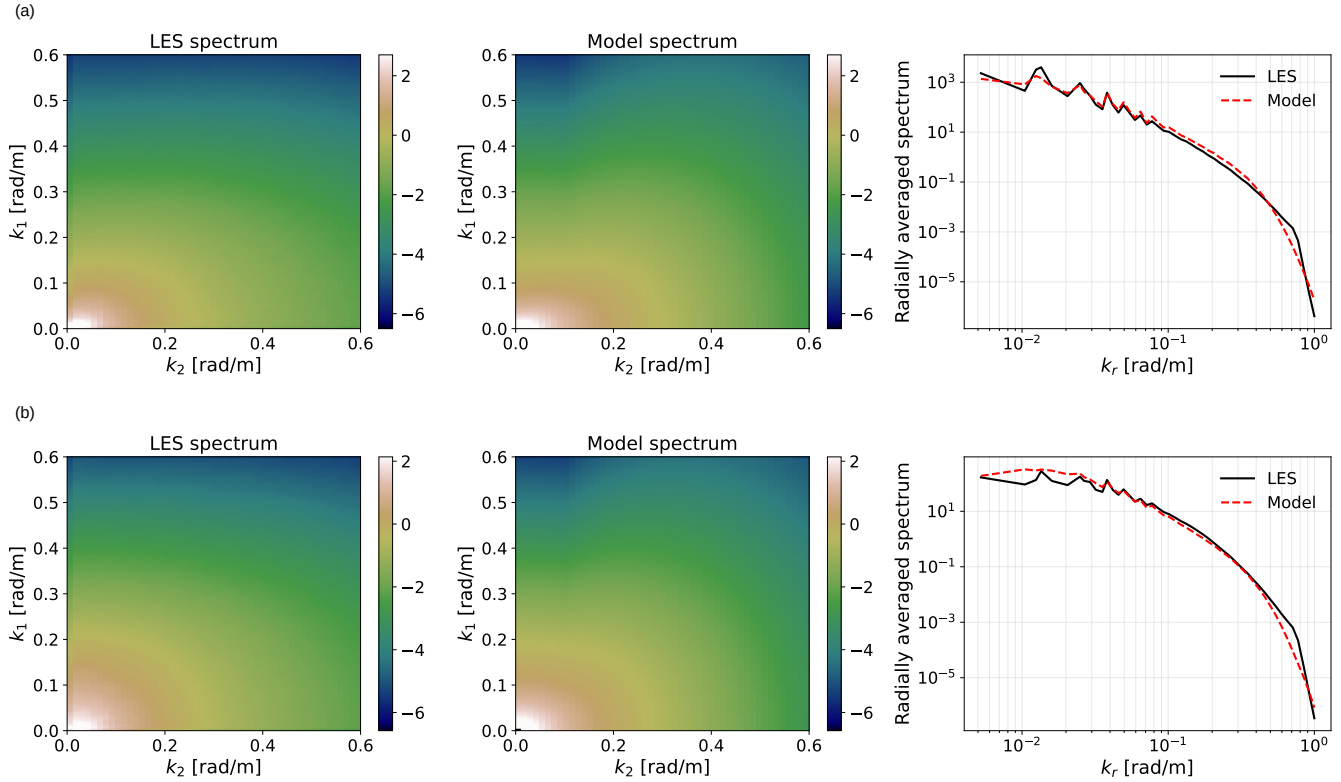


Figure 3. ABL case: comparison of the wavenumber spectra obtained from ABL LES data and the anisotropic spectrum model for (a) the streamwise and (b) the wall-normal velocity components.

low-wavenumber wall-normal spectrum, where the modeled curve is smoother and roughly a factor of two higher than the first LES bins. At these scales, the measured spectrum is strongly influenced by wake-scale motions and by the finite spatial extent of the sampled plane. In the WT-LTI case shown in Fig. 5, the low-wavenumber peaks are more pronounced, particularly in the streamwise component, where an isolated maximum appears near $k_r \approx 1.3 \times 10^{-2} \text{ rad m}^{-1}$. The model does not reproduce this individual peak, but it captures the surrounding spectral level and the subsequent decay. The wall-normal spectrum of this case differs qualitatively from the other two configurations: its maximum occurs at a finite wavenumber, $k_r \approx 3 \times 10^{-2} \text{ rad m}^{-1}$, and the LES spectrum decreases again toward the largest resolved scales. The model reproduces this non-monotonic shape and the position of the maximum, with a mild overestimation of its level and a slightly steeper decay over the last bins. Small deviations near the final wavenumber bins should be interpreted cautiously because this region is affected by the LES numerical dissipation, the finite grid resolution, and the explicit dissipative correction used in the model.

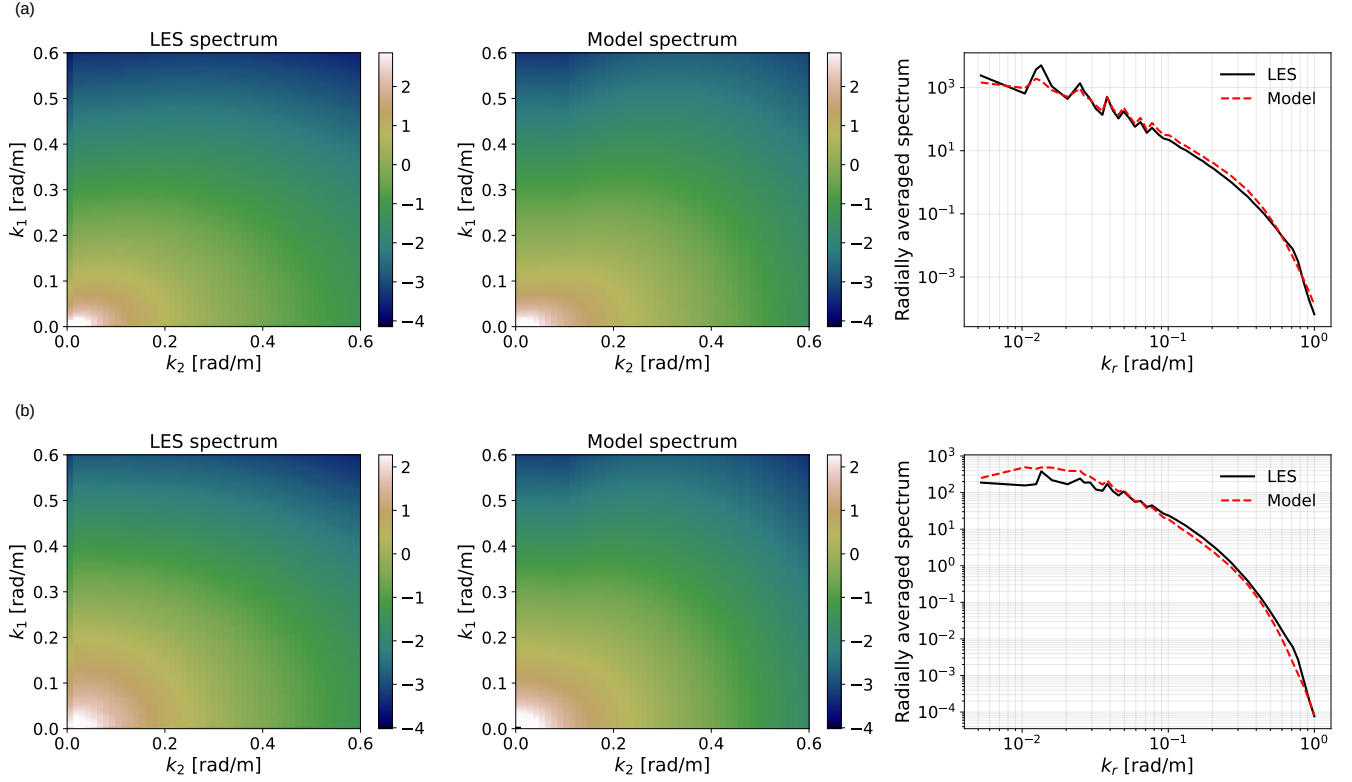


Figure 4. WT-HTI case: comparison of the wavenumber spectra obtained from LES data and the anisotropic spectrum model for (a) the streamwise and (b) the wall-normal velocity components.

6.3 Scale-dependent anisotropy and mapping admissibility

Figs. 6, 7, and 8 show the fitted principal stretching factors $\mu_1(\mathbf{k})$ and $\mu_2(\mathbf{k})$, the invariant anisotropy measure $\gamma(\mathbf{k})$, the signed Jacobian determinant $J_s(\mathbf{k})$, and the two Bernstein decay surfaces $\Psi_1(\hat{q}_1, \hat{q}_2)$ and $\Psi_2(\hat{q}_1, \hat{q}_2)$. The first four fields are displayed in the physical (k_y, k_x) plane, where k_1 denotes the streamwise wavenumber k_x and k_2 the spanwise wavenumber k_y , while the Bernstein surfaces are shown in the normalized coordinates $(\hat{q}_1, \hat{q}_2)$ associated with these two directions. These fields provide a more direct physical interpretation of the fitted spectral deformation than the radially averaged spectra alone. A consistent result across the three cases is that the dominant deformation is associated with μ_1 , whereas μ_2 remains very close to unity. The fitted departure of μ_2 from unity does not exceed about 10^{-6} in any of the displayed cases. Consequently, the invariant

$$\gamma(\mathbf{k}) = \frac{1}{2} \ln \left[\frac{\mu_1(\mathbf{k})}{\mu_2(\mathbf{k})} \right] \quad (129)$$

is controlled almost entirely by the first principal stretching direction, and the numerical values of γ follow $\frac{1}{2} \ln \mu_1$ to within the plotting accuracy. Thus, although the formulation permits a fully bidirectional deformation, the present data support an

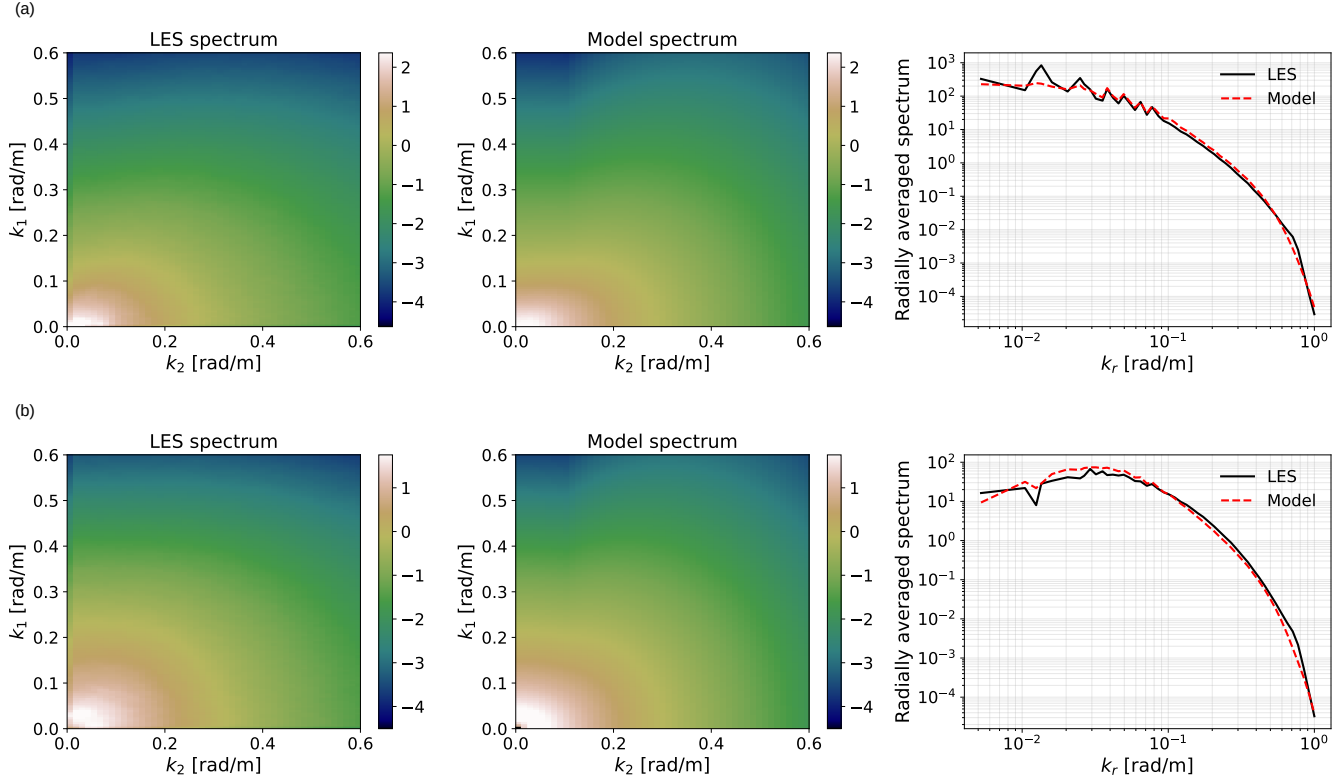


Figure 5. WT-LTI case: comparison of the wavenumber spectra obtained from LES data and the anisotropic spectrum model for (a) the streamwise and (b) the wall-normal velocity components.

effectively one-dominant-direction anisotropy. This is a useful data-driven result, where the optimization does not impose a second deformation when it is not required by the measured spectra.

A second feature common to the three configurations is the orientation of the fitted deformation. In all cases μ_1 , γ , and J_s are essentially independent of k_x and vary only with k_y , so that their isolines are straight lines of constant spanwise wavenumber rather than circular arcs. The anisotropy is largest in a band at low k_y , relaxes monotonically as k_y increases, and reaches the isotropic limit $\mu_1 = 1$, $\gamma = 0$ at the upper end of the displayed range. Anisotropy therefore remains finite when k_1 is large but k_2 remains small, which is precisely the residual edge anisotropy permitted by the bivariate formulation. This behavior could not be represented by a purely radial decay function depending only on $\sqrt{k_1^2 + k_2^2}$.

The ABL reference case exhibits the strongest deformation of the three configurations. Its first stretching factor decreases approximately from 1.58 at the most anisotropic scales to unity near $k_y \approx 0.6 \text{ rad m}^{-1}$, and the corresponding invariant anisotropy spans approximately 0 to 0.23. The plateau of maximum stretching extends to $k_y \approx 0.15 \text{ rad m}^{-1}$, beyond which the transition toward isotropy is smooth and monotone. The fitted surface Ψ_1 changes almost exclusively with q_2 and is nearly independent of q_1 , so that the relaxation of the anisotropy is governed by the second wavenumber coordinate alone. The WT-HTI case, shown

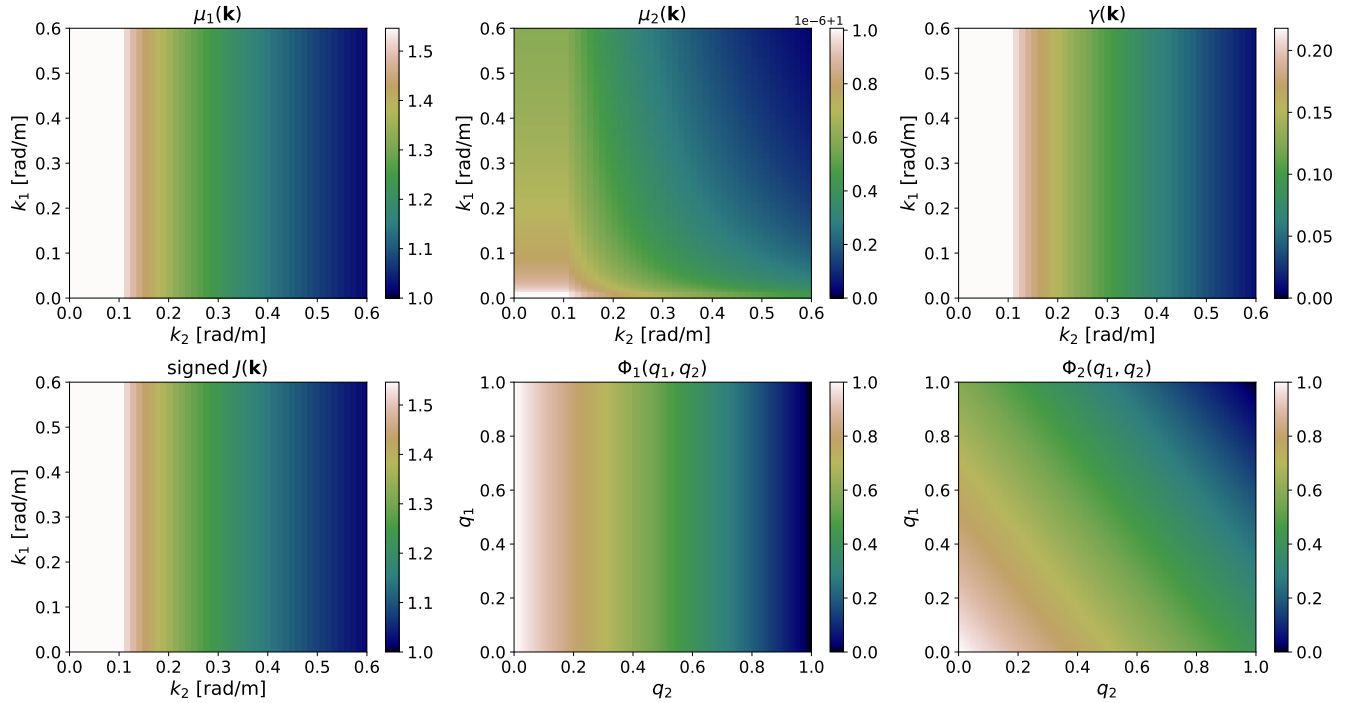


Figure 6. ABL case: Fitted scale-dependent anisotropy diagnostics, including the principal stretching factors, the invariant anisotropy measure, the signed Jacobian determinant, and the Bernstein decay surfaces.

in Fig. 7, exhibits the weakest anisotropy among the three displayed cases. Its first stretching factor varies approximately from 1.43 down to unity, while γ reaches at most about 0.18, and the departure from the isotropic state begins at a slightly smaller spanwise wavenumber, $k_y \approx 0.1 \text{ rad m}^{-1}$, than in the ABL case. The surface Ψ_1 again depends predominantly on q_2 , with only a slight tilt indicating a weak residual dependence on q_1 . The reduced magnitude of γ relative to the WT-LTI case is consistent with stronger ambient turbulence promoting wake mixing and weakening the persistence of directionally organized large-scale motions.

The WT-LTI case retains a stronger deformation than WT-HTI, with μ_1 reaching values close to 1.48 and γ approaching approximately 0.20 at the most anisotropic scales, so that the three configurations are ordered as ABL, WT-LTI, and WT-HTI by decreasing anisotropy amplitude. The two wake cases also differ from the ABL reference in the structure of the secondary fields. Whereas the ABL Ψ_2 surface varies mainly with q_2 , the WT-HTI and WT-LTI Ψ_2 surfaces display clearly anti-diagonal isolines and therefore depend jointly on q_1 and q_2 ; the same coupling appears in the corner-shaped structure of μ_2 near small k_y and small k_x in both wake cases. This suggests that wake-generated structures retain a coupled streamwise–spanwise scale dependence that is absent from the shear-driven ABL reference. The fitted Ψ_2 surfaces should nevertheless be interpreted with care. Although they show smooth and monotone variation in normalized wavenumber space, their contribution to the physical mapping is multiplied by $\mu_{2,\text{max}} - 1$, which is of order 10^{-6} in all three cases. Their detailed shape is therefore weakly influ-

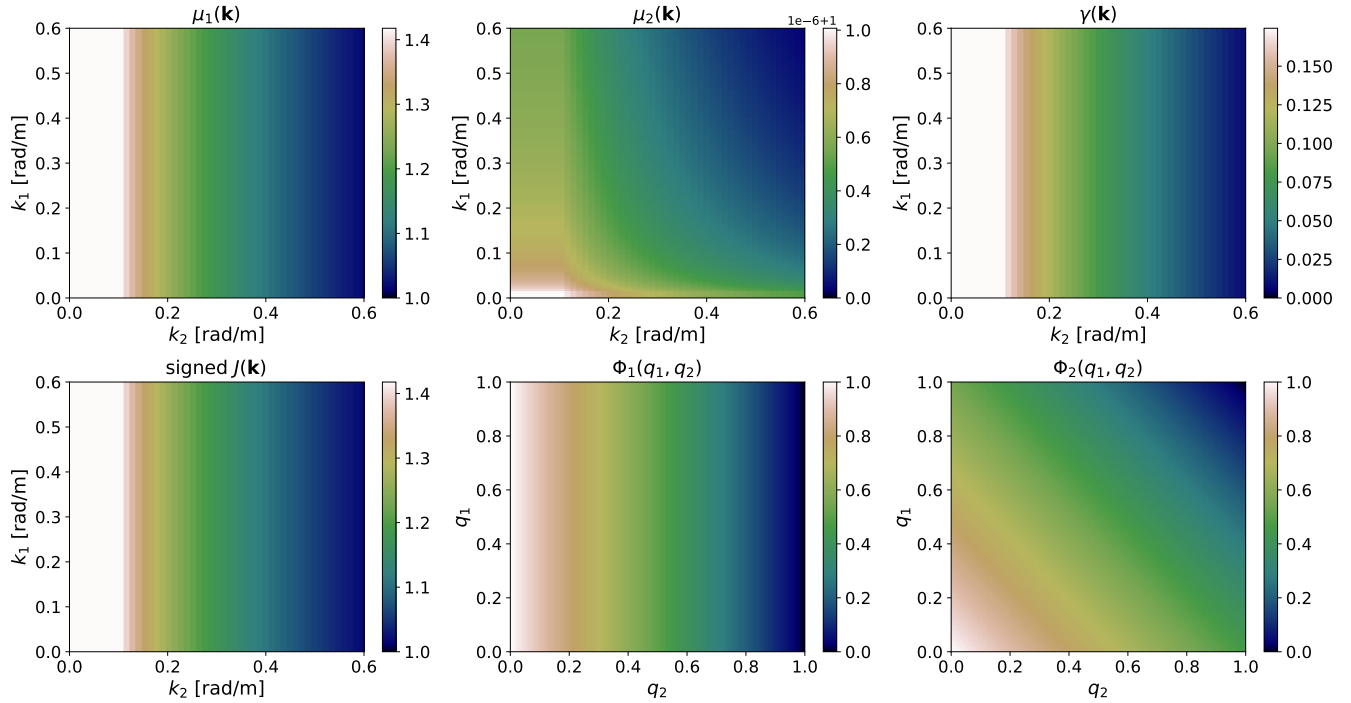


Figure 7. WT-HTI case: Fitted scale-dependent anisotropy diagnostics, including the principal stretching factors, the invariant anisotropy measure, the signed Jacobian determinant, and the Bernstein decay surfaces.

tial and potentially weakly identifiable, and the coupling described above should not be read as a dynamically significant effect.

825 The physically meaningful anisotropy in the present fits is primarily contained in μ_1 , Ψ_1 , and the resulting γ field. The signed Jacobian remains positive throughout the displayed wavenumber domain for all three configurations, with values bounded between unity in the isotropic limit and the maximum of μ_1 at the largest scales. Hence, the fitted spectral transformation is locally orientation preserving and does not fold the wavenumber plane. The spatial variation of $J_s(\mathbf{k})$ coincides with that of μ_1 to within the resolution of the colorbars, because μ_2 is nearly unity, and the transition of J_s toward its isotropic value is smooth

830 and monotone in k_y in every case. This confirms that the additional scale dependence and cross-coordinate coupling have been introduced without violating the local invertibility of the spectral mapping. The comparison between the WT-HTI and WT-LTI cases therefore indicates that ambient turbulence intensity affects not only the total spectral energy, but also the geometry of its distribution in wavenumber space. The WT-HTI case is closer to an isotropic state, whereas the WT-LTI case preserves a stronger large-scale anisotropy. The ABL reference exhibits the largest anisotropy of the three, although its physical origin is

835 associated mainly with boundary-layer shear and large-scale organization rather than turbine-generated wake dynamics.

Next, we compare three model formulations to assess the contribution of the anisotropy representation, including an isotropic baseline, a prescribed constant-anisotropy model, and the proposed scale-dependent anisotropy model. The three models are

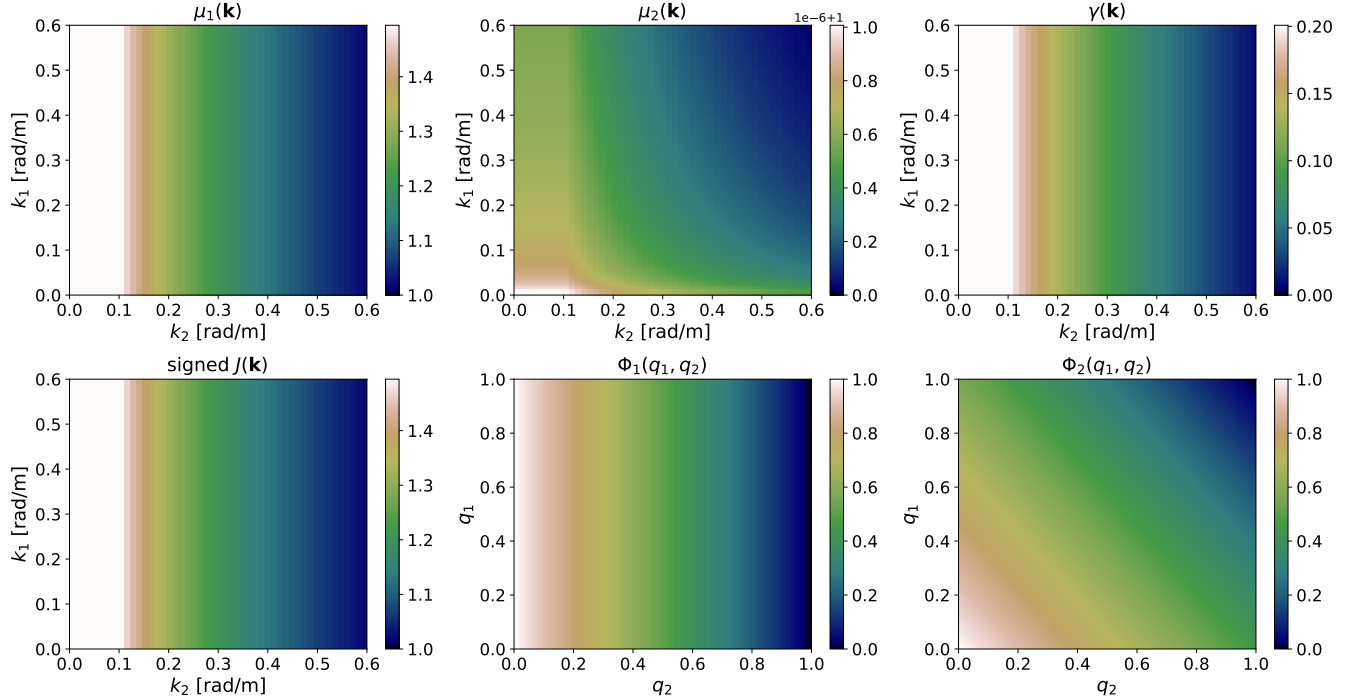


Figure 8. WT-LTI case: Fitted scale-dependent anisotropy diagnostics, including the principal stretching factors, the invariant anisotropy measure, the signed Jacobian determinant, and the Bernstein decay surfaces.

defined as

$$\mathcal{C}_1 : \quad \mu_1 = \mu_2 = 1, \quad (130)$$

$$840 \quad \mathcal{C}_2 : \quad \mu_1 = 1.5, \quad \mu_2 = 1.2, \quad (131)$$

$$\mathcal{C}_3 : \quad \mu_1 = \mu_1(k_1, k_2), \quad \mu_2 = \mu_2(k_1, k_2). \quad (132)$$

The models are evaluated using the combined log-spectral root-mean-square error (log-RMSE) of the u - and w -velocity spectra, with lower values indicating better agreement with the reference spectra. The results are summarized in Table 5. Introducing a fixed anisotropy ($\mathcal{C}_2$) already produces a substantial improvement over the isotropic model for all three data sets, reducing the log-RMSE from 1.655 to 0.850 for ABL, from 0.931 to 0.422 for WT-HTI, and from 1.097 to 0.507 for WT-LTI. Allowing the anisotropy to vary with (k_1, k_2) further reduces the error to 0.455, 0.335, and 0.270, respectively. Relative to the isotropic baseline, $\mathcal{C}_3$ therefore reduces the spectral error by approximately 73% for ABL, 64% for WT-HTI, and 75% for WT-LTI. The additional improvement over the fixed-anisotropy model is particularly pronounced for ABL and WT-LTI, indicating that their anisotropy cannot be represented adequately by a single scale-independent correction. For WT-HTI, the improvement is more moderate, suggesting that a larger fraction of its anisotropic structure can already be captured by the prescribed constant scaling.



Table 5. Comparison of the isotropic, fixed-anisotropy, and scale-dependent anisotropy models based on the combined log-RMSE of the u and w spectra.

Model	Anisotropy	Params.	ABL	WT-HTI	WT-LTI
$\mathcal{C}_1$	Isotropic	7	1.655	0.931	1.097
$\mathcal{C}_2$	Fixed	7	0.850	0.422	0.507
$\mathcal{C}_3$	Scale-dependent	29	0.455	0.335	0.270

6.4 Non-Gaussian synthetic velocity fields

Figs. 9–11 compare the normalized LES probability density functions with Gaussian, moment-matched Hermite, and rank-based synthetic fields. These comparisons assess a statistical property that is not represented by the fitted spectrum alone. The planar spectrum determines the second-order spatial statistics, but it does not uniquely determine the one-point probability distribution unless Gaussianity is assumed. The Gaussian realizations reproduce the central part of distributions that are close to normal, but they systematically underestimate probabilities in the far tails whenever the LES fluctuations are intermittent. This discrepancy is expected because the Gaussian synthesis contains no independent control of skewness or kurtosis. The Hermite transformation substantially improves the tail representation by matching the LES skewness and kurtosis, while the rank-based transformation additionally incorporates the complete empirical marginal distribution.

For the ABL case in Fig. 9, the streamwise velocity distribution is relatively close to Gaussian over its central range, although departures become visible in the low-probability tails. The wall-normal fluctuations show a sharper central peak and heavier tails, indicating a stronger departure from Gaussianity. The Hermite and rank-based realizations reproduce these features more accurately than the Gaussian realization. Minor differences remain near the sharp PDF maximum and at the most extreme sampled amplitudes, where histogram estimation is sensitive to the finite number of realizations. The WT-HTI distributions in Fig. 10 remain moderately non-Gaussian. The Gaussian model captures the dominant core of both components but decays too rapidly in the outer tails. Both non-Gaussian transformations follow the LES more closely over the resolved probability range. The relatively moderate departure from Gaussianity is consistent with enhanced turbulent mixing in the WT-HTI, which reduces the persistence of isolated coherent events and produces more statistically mixed fluctuations.

The strongest non-Gaussian behavior is observed for the WT-LTI in Fig. 11. The streamwise component exhibits probabilities well above the standard normal distribution at large normalized amplitudes. The wall-normal distribution combines a pronounced central peak with broad tails, providing a clear signature of intermittency. The Gaussian realization cannot represent this simultaneous peakedness and heavy-tailed behavior. The Hermite model gives a major improvement, while the rank-based realization follows the LES distribution over nearly the complete plotted range. This result is consistent with a low-turbulence wake in which coherent shear-generated structures and localized strong fluctuations remain less disrupted by the surrounding turbulence. The close agreement of the rank-based curves with the LES PDFs should be interpreted as a consistency check rather than an independent prediction of the marginal distribution. The empirical LES distribution is used to

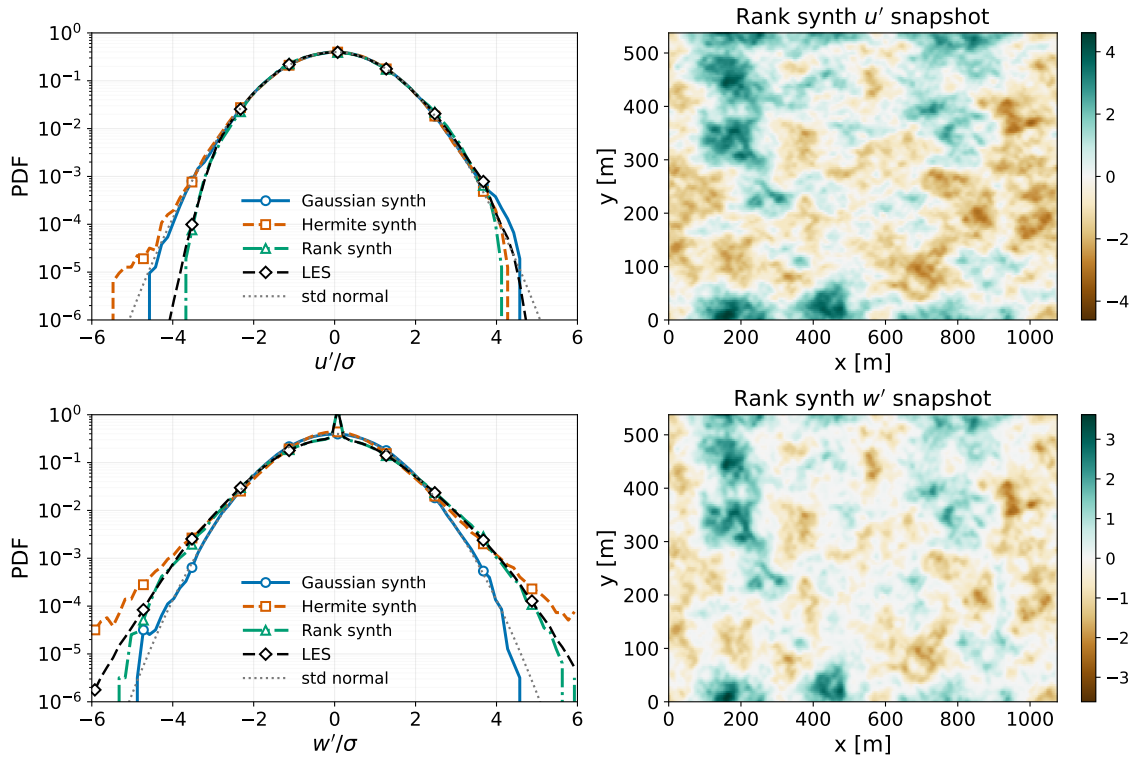


Figure 9. ABL case: Probability density functions of the normalized streamwise and wall-normal velocity fluctuations, u'/σ_u and w'/σ_w , comparing LES data with Gaussian, Hermite, and rank-based synthetic realizations. The right panels show representative rank-based synthetic snapshots of the generated ABL flow fields for u' and w' .

construct the rank transformation, and close PDF agreement is therefore expected by design. The significant result is that this marginal transformation can be combined with the fitted spatial spectrum, allowing the generated fields to reproduce both the prescribed second-order organization and the observed one-point non-Gaussianity. By contrast, the Hermite model uses only two non-Gaussian moments, so its agreement with the remaining shape of the PDF provides a more restrictive assessment of the low-order translation approximation. The representative snapshots in the right panels contain spatially correlated patches rather than uncorrelated pointwise noise. Their characteristic dimensions and directional organization are inherited from the fitted planar spectra, whereas the occurrence and amplitude of the extreme values are controlled by the non-Gaussian marginal transformation. The snapshots therefore demonstrate the practical connection between the spectral model and synthetic turbulence generation, where the spectral fit determines where spatial coherence occurs, while the translation process determines how frequently unusually large positive or negative fluctuations occur.

Figure 12 provides a representative physical-space comparison between the LES velocity fluctuations and the rank-based synthetic fields at two wake-centerline locations for the WT-HTI case. At $x/D = 1$, the LES streamwise fluctuation exhibits pronounced, slowly varying excursions that persist over hundreds of samples, indicative of energetic, temporally coherent

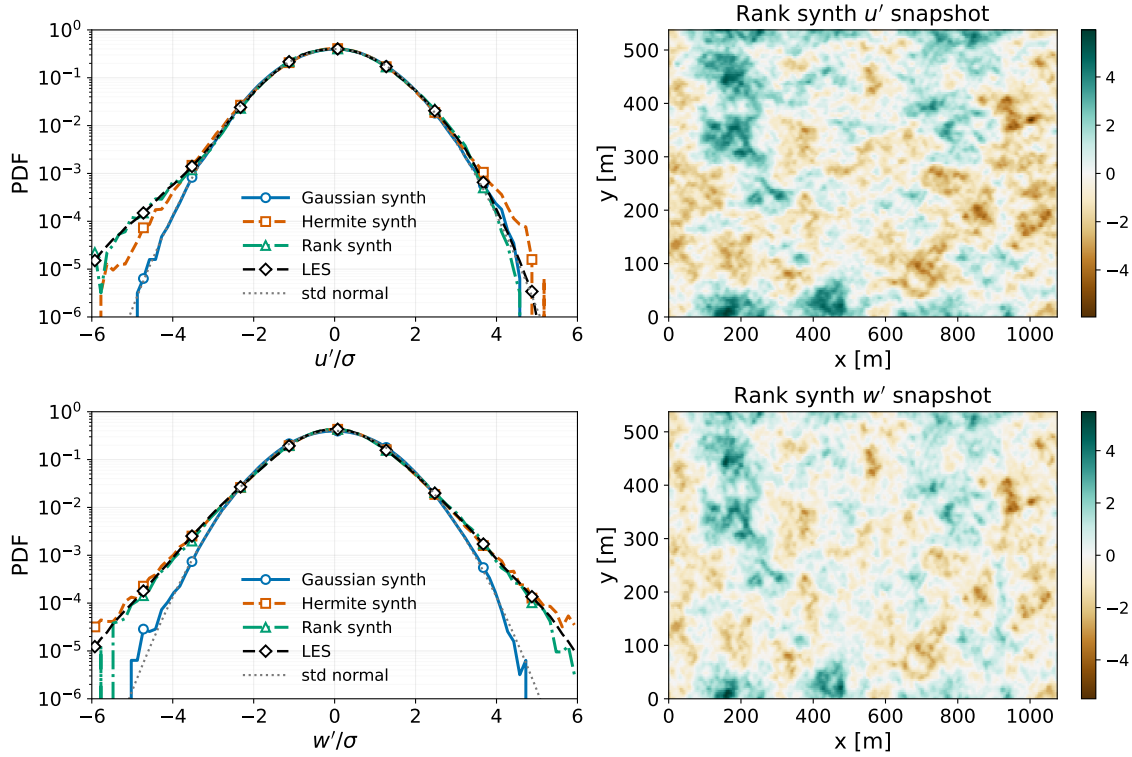


Figure 10. WT-HTI case: Probability density functions of the normalized streamwise and wall-normal velocity fluctuations, u'/σ_u and w'/σ_w , comparing LES data with Gaussian, Hermite, and rank-based synthetic realizations. The right panels show representative rank-based synthetic snapshots of the generated wake flow fields for u' and w' .

motions in the near wake. Farther downstream, at $x/D = 5$, the streamwise signal varies more rapidly, attains smaller amplitudes, and contains fewer sustained positive or negative excursions, consistent with the progressive disruption of the near-wake structures and stronger turbulent mixing. The wall-normal component displays weaker low-frequency organization than the streamwise component at both stations; in the displayed segment its fluctuations become more intermittent at $x/D = 5$, where several relatively strong events are observed, including a large negative excursion near the end of the record. The rank-based realizations reproduce the overall order of magnitude and the resolved range of the LES fluctuations for both velocity components. This follows from the empirical quantile transformation, which maps the underlying Gaussian field onto the sampled LES marginal distribution, so that any departure from Gaussianity present in that distribution is inherited by the synthetic field by construction rather than being imposed a posteriori. The gray sequences, however, show no correlation between successive displayed samples. This is a direct consequence of the construction, where each displayed value is extracted from an independent spatial realization, and the fitted two-dimensional spectrum prescribes spatial rather than temporal correlations. The comparison therefore constrains the amplitude and marginal statistics of the synthetic field, not its temporal organization, which the present model does not attempt to represent. The synthetic envelope is also nearly identical at the two downstream

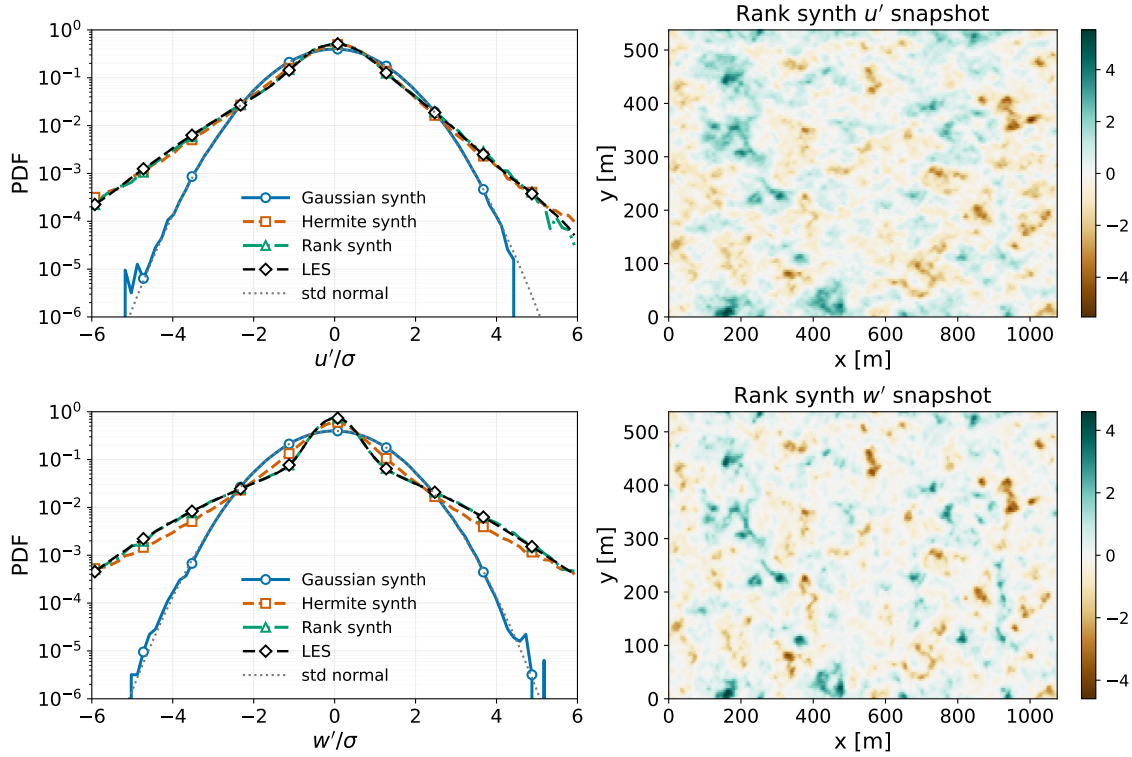


Figure 11. WT-LTI case: Probability density functions of the normalized streamwise and wall-normal velocity fluctuations, u'/σ_u and w'/σ_w , comparing LES data with Gaussian, Hermite, and rank-based synthetic realizations. The right panels show representative rank-based synthetic snapshots of the generated wake flow fields for u' and w' .

positions, whereas the LES amplitude varies appreciably between them. As a result, the synthetic spread is somewhat too
 905 broad relative to the local LES signal for w' at $x/D = 1$, while the largest near-wake streamwise excursions at the same station
 lie outside the synthetic range. Both discrepancies arise from the same source that the empirical rank transformation and the
 fitted spectrum are constructed from the complete analyzed LES plane rather than from the local statistics at each downstream
 position.

7 Discussion and conclusions

910 This study introduced a physically constrained framework for modeling two-dimensional wavenumber spectra in atmospheric
 boundary-layer and wind-turbine wake turbulence. The proposed formulation represents anisotropy through a bidirectional,
 scale-dependent tensorial remapping of spectral space. Unlike models based on a constant stretching factor or a correction ap-
 plied along only one wavenumber direction, the present approach allows the spectral deformation to vary continuously across
 the full (k_1, k_2) plane. It can therefore describe strong anisotropy at the energy-containing scales and its progressive relaxation

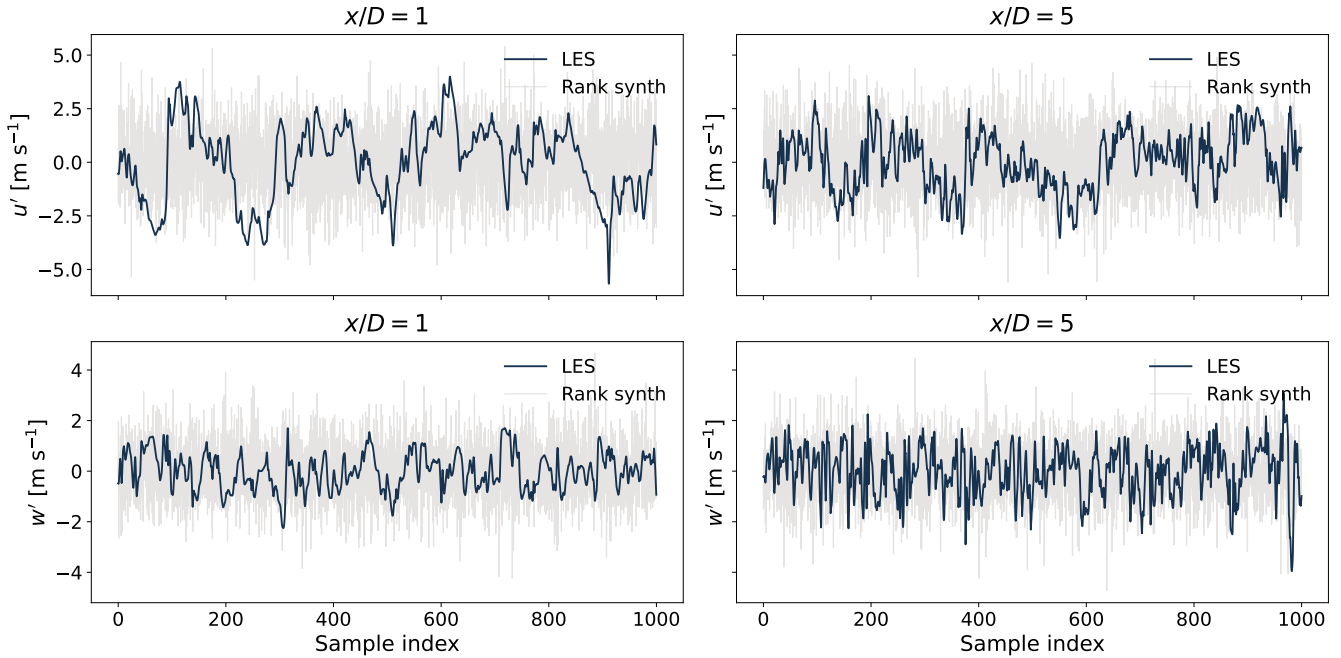


Figure 12. Comparison of the streamwise and wall-normal velocity fluctuations, u' and w' , obtained from the LES and the rank-based synthetic turbulence at the wake-centerline locations $x/D = 1$ and $x/D = 5$ for the WT-HTI case.

915 toward isotropy at smaller scales. The scale dependence of the remapping is represented by monotone bivariate Bernstein surfaces, which provide a smooth and physically admissible description of the anisotropy decay. Because the stretching factors may depend on both wavenumber coordinates, the complete two-dimensional Jacobian contains cross-coordinate coupling terms that do not appear in simpler one-dimensional transformations. Including this Jacobian is essential for consistently transforming the spectral density and preventing nonphysical folding of the wavenumber plane. In all investigated cases, the fitted

920 signed Jacobian remained positive over the modeled spectral domain, confirming that the resulting transformations were locally orientation preserving. The model parameters were identified through a staged, multi-resolution optimization procedure that separates the estimation of the isotropic spectral backbone from the anisotropic deformation before performing a final joint refinement. The logarithmic spectral mismatch constrains the spectral shape over several orders of magnitude, while the energy-consistency term preserves the measured spectral variances. The Jacobian and Bernstein-surface penalties further en-

925 sure admissibility and suppress nonphysical or excessively irregular anisotropy fields. Comparisons with the LES data show that the model reproduces the dominant directional organization and broadband decay of the streamwise and wall-normal spectra for the ABL reference, high-turbulence-intensity wake, and low-turbulence-intensity wake cases, following the measured spectral envelope over the full resolved wavenumber interval and across six to seven orders of magnitude in spectral density. Agreement across the complete two-dimensional spectral distributions provides stronger evidence of model performance than

930 radial agreement alone, since radial averaging removes the directional information that the model is intended to recover. The



remaining differences occur mainly at the lowest resolved wavenumbers, where the LES spectra contain narrow peaks and troughs associated with a limited number of Fourier modes, coherent structures, and spatial inhomogeneity. The smooth parametric formulation should therefore be interpreted as a model of the statistical spectral envelope rather than a deterministic reconstruction of individual coherent spectral peaks. Although the formulation permits fully bidirectional deformation, the fitted results indicate that the dominant anisotropy in the present cases is carried almost entirely by a single principal stretching direction. The second stretching factor remained at its lower bound in all three fits, and the associated decay surface is consequently weakly identifiable. The relaxation toward isotropy is likewise governed almost exclusively by the second spectral coordinate, so that the fitted anisotropy fields vary with the spanwise wavenumber and remain nearly independent of the streamwise one. This is a data-driven outcome rather than an imposed restriction, and it illustrates the residual edge anisotropy permitted by the bivariate construction, which could not be represented by a purely radial decay function. It also implies that the cross-coupling terms retained in the Jacobian, although necessary for a consistent general formulation, are numerically small for the present datasets.

The three configurations differ mainly in the amplitude and in the shape of the anisotropy decay. The ABL reference exhibits the largest large-scale deformation and an almost purely quadratic decay profile, which keeps the anisotropy active across most of the energy-containing range before it collapses near the isotropization scale; its spectral organization is associated primarily with boundary-layer shear and large-scale atmospheric structures rather than turbine-generated wake dynamics. Between the two wake cases, the low-turbulence-intensity wake retains the stronger scale-dependent deformation, indicating that coherent shear-generated motions persist over a broader range of scales when ambient turbulence is weak, whereas the high-turbulence-intensity wake exhibits the weakest anisotropy, consistent with stronger ambient mixing and more rapid disruption of directionally organized wake structures. Both wake cases share a mixed linear–quadratic decay profile and therefore isotropize earlier and more gradually than the ABL reference. Ambient turbulence intensity thus affects not only the total spectral energy but also the geometry of its distribution in wavenumber space. The fitted spectra also provide a direct basis for generating synthetic turbulent velocity fields. Gaussian spectral synthesis reproduces the prescribed second-order spatial organization, including the characteristic dimensions and directional alignment of the turbulent structures. The subsequent Hermite and rank-based translation procedures introduce non-Gaussian one-point statistics while retaining the fitted spatial structure to a controlled accuracy. The Hermite transformation substantially improves the representation of skewness, kurtosis, and probability-density tails using only low-order moments, whereas the rank-based transformation reproduces the complete empirical marginal distribution by construction. The particularly strong non-Gaussianity observed in the low-turbulence-intensity wake further illustrates the persistence of intermittent and coherent wake-generated fluctuations under weak ambient mixing. Future work should extend the formulation to include cross-spectral terms and the complete velocity spectral tensor, spatially evolving anisotropy at different downstream locations, non-aligned or scale-dependent principal directions, and a wider range of atmospheric stability and turbine operating conditions. Improved identifiability of the second stretching direction and of the transition wavenumbers, several of which were bound-limited in the present fits, would also be needed before the secondary deformation can be given a physical interpretation. Such developments would support the integration of the model into engineering wake simulations, inflow-generation methods, aeroelastic load calculations, and reduced-order representations of wind-farm turbulence.



Code and data availability. The data and code supporting the findings of this study are available from the corresponding author upon reasonable request. The large-eddy simulation data used in this study are subject to the data-sharing conditions of the originating research project and institution.

Author contributions. SS implemented and executed the proposed model, carried out the numerical analyses, investigated the optimization aspects of the framework, performed sensitivity analyses, and contributed to the interpretation and revision of the manuscript. MG contributed to the methodological development, supervised the research, provided guidance on the analysis and interpretation of the results, and reviewed and edited the manuscript. NA contributed substantially to the conceptual development of the study and the underlying turbulence and spectral-modeling framework, provided domain expertise on wind-turbine wake turbulence, supported the interpretation of the LES data and spectral results, and contributed to the review and revision of the manuscript. All authors discussed the results and approved the final version of the manuscript.

Competing interests. The authors declare that they have no conflict of interest.

Financial support. N.A. acknowledges the Helmholtz Information and Data Science Academy (HIDA)-Norway mobility program for providing financial support enabling a short-term research stay at the Western Norway University of Applied Sciences.



References

- 980 Ali, N. and Cal, R. B.: Scale evolution, intermittency and fluctuation relations in the near-wake of a wind turbine array, *Chaos, Solitons & Fractals*, 119, 215–229, <https://doi.org/10.1016/j.chaos.2018.12.018>, 2019.
- Ali, N., Kadum, H. F., and Cal, R. B.: Focused-based multifractal analysis of the wake in a wind turbine array utilizing proper orthogonal decomposition, *Journal of Renewable and Sustainable Energy*, 8, <https://doi.org/10.1063/1.4968032>, 2016.
- Ali, N., Hamilton, N., and Calaf, M. and Cal, R. B.: Turbulence kinetic energy budget and conditional sampling of momentum, scalar, and
 985 intermittency fluxes in thermally stratified wind farms, *Journal of Turbulence*, 20, 32–63, <https://doi.org/10.1080/14685248.2018.1564831>, 2019.
- Ali, N., Gatti, D., and Kornev, N.: Tailoring anisotropic synthetic inflow turbulence generator for wind turbine wake simulations, *Journal of Renewable and Sustainable Energy*, 16, <https://doi.org/10.1063/5.0217802>, 2024.
- Alrewny, A., Vandewalle, S., and Meyers, J.: Large-eddy simulation-based reconstruction of turbulence in a neutral boundary layer using
 990 spectral-tensor regularization, *Journal of Fluid Mechanics*, 981, A28, <https://doi.org/10.1017/jfm.2024.92>, 2024.
- Cal, R. B., Lebrón, J., Castillo, L., Kang, H. S., and Meneveau, C.: Experimental study of the horizontally averaged flow structure in a model wind-turbine array boundary layer, *Journal of Renewable and Sustainable Energy*, 2, 013 106, <https://doi.org/10.1063/1.3289735>, 2010.
- Chandran, D., Baidya, R., Monty, J. P., and Marusic, I.: Two-dimensional energy spectra in high-Reynolds-number turbulent boundary layers, *Journal of Fluid Mechanics*, 826, R1, <https://doi.org/10.1017/jfm.2017.359>, 2017.
- 995 Cheng, Y., Li, Q., Argentini, S., Sayde, C., and Gentine, P.: A model for turbulence spectra in the equilibrium range of the stable atmospheric boundary layer, *Journal of Geophysical Research: Atmospheres*, 125, e2019JD032 191, <https://doi.org/10.1029/2019JD032191>, 2020.
- Cheyne, E., Jakobsen, J. B., and Reuder, J.: Velocity Spectra and Coherence Estimates in the Marine Atmospheric Boundary Layer, *Boundary-Layer Meteorology*, 169, 429–460, <https://doi.org/10.1007/s10546-018-0382-2>, 2018.
- Chung, D., Marusic, I., Monty, J. P., Vallikivi, M., and Smits, A. J.: On the universality of inertial energy in the log layer of turbulent boundary
 1000 layer and pipe flows, *Experiments in Fluids*, 56, 1–10, <https://doi.org/10.1007/s00348-015-1994-7>, 2015.
- Davidson, P. A., Nickels, T. B., and Krogstad, P.-Å.: The logarithmic structure function law in wall-layer turbulence, *Journal of Fluid Mechanics*, 550, 51–60, <https://doi.org/10.1017/S0022112005008001>, 2006.
- de Kat, R. and Ganapathisubramani, B.: Frequency-wavenumber mapping in turbulent shear flows, *Journal of Fluid Mechanics*, 783, 166–190, <https://doi.org/10.1017/jfm.2015.558>, 2015.
- 1005 Del Álamo, J. C. and Jiménez, J.: Estimation of turbulent convection velocities and corrections to Taylor’s approximation, *Journal of Fluid Mechanics*, 640, 5, <https://doi.org/10.1017/S0022112009991029>, 2009.
- Favre, A. J.: Review on space-time correlations in turbulent fluids, *Journal of Applied Mechanics*, 32, 241–257, <https://doi.org/10.1115/1.3625792>, 1965.
- Glegg, S. and Devenport, W.: *Aeroacoustics of low Mach number flows: fundamentals, analysis, and measurement*, Academic Press, <https://doi.org/https://doi.org/10.1016/B978-0-12-809651-2.00008-4>, 2017.
- 1010 Gouder, K., Milne, I. A., and Graham, M.: Turbulence distortion and blockage in the induction zone of a horizontal axis turbine, *Journal of Fluid Mechanics*, 1000, A25, <https://doi.org/10.1017/jfm.2024.713>, 2024.
- Grigoriu, M.: *Applied Non-Gaussian Processes: Examples, Theory, Simulation, Linear Random Vibration, and MATLAB Solutions*, Prentice Hall, Englewood Cliffs, NJ, ISBN 978-0-13-367095-0, 1995.



- 1015 Grigoriu, M.: Simulation of Stationary Non-Gaussian Translation Processes, *Journal of Engineering Mechanics*, 124, 121–126, [https://doi.org/10.1061/\(ASCE\)0733-9399\(1998\)124:2\(121\)](https://doi.org/10.1061/(ASCE)0733-9399(1998)124:2(121)), 1998.
- Hunt, J. C. R. and Morrison, J. F.: Eddy structure in turbulent boundary layers, *European Journal of Mechanics-B/Fluids*, 19, 673–694, [https://doi.org/10.1016/S0997-7546\(00\)00129-1](https://doi.org/10.1016/S0997-7546(00)00129-1), 2000.
- Kelly, M. and Wyngaard, J. C.: Two-dimensional spectra in the atmospheric boundary layer, *Journal of the Atmospheric Sciences*, 63, 3066–3070, <https://doi.org/10.1175/JAS3769.1>, 2006.
- 1020 Kosović, B., Basu, S., Berg, J., Berg, L. K., Haupt, S. E., Larsén, X. G., Peinke, J., Stevens, R. J. A. M., Veers, P., and Watson, S.: Impact of atmospheric turbulence on performance and loads of wind turbines: knowledge gaps and research challenges, *Wind Energy Science*, 11, 509–555, <https://doi.org/10.5194/wes-11-509-2026>, 2026.
- Kraichnan, R. H.: Inertial Ranges in Two-Dimensional Turbulence, *The Physics of Fluids*, 10, 1417–1423, <https://doi.org/10.1063/1.1762301>, 1967.
- 1025 Larsén, X. G., Vincent, C. L., and Larsen, S. E.: Spectral Structure of Mesoscale Winds over the Water, *Quarterly Journal of the Royal Meteorological Society*, 139, 685–700, <https://doi.org/10.1002/qj.2003>, 2013.
- Larsén, X. G., Larsen, S. E., and Petersen, E. L.: Full-Scale Spectrum of Boundary-Layer Winds, *Boundary-Layer Meteorology*, 159, 349–371, <https://doi.org/10.1007/s10546-016-0129-x>, 2016.
- 1030 Lindborg, E.: Can the Atmospheric Kinetic Energy Spectrum Be Explained by Two-Dimensional Turbulence?, *Journal of Fluid Mechanics*, 388, 259–288, <https://doi.org/10.1017/S0022112099004851>, 1999.
- Michelsen, J. A.: Block structured multigrid solution of 2D and 3D elliptic PDE's, Department of Fluid Mechanics, Technical University of Denmark, 1994.
- Morrison, W. R. B. and Kronauer, R. E.: Structural similarity for fully developed turbulence in smooth tubes, *Journal of Fluid Mechanics*, 39, 117–141, <https://doi.org/10.1017/S0022112069002072>, 1969.
- 1035 Narita, Y.: Spectral moments for the analysis of frequency shift, broadening, and wavevector anisotropy in a turbulent flow, *Earth, Planets and Space*, 69, 1–10, <https://doi.org/10.1186/s40623-017-0658-7>, 2017.
- Perry, A. E. and Chong, M. S.: On the mechanism of wall turbulence, *Journal of Fluid Mechanics*, 119, 173–217, <https://doi.org/10.1017/S0022112082001311>, 1982.
- 1040 Placidi, M., Hancock, P. E., and Hayden, P.: Wind turbine wakes: experimental investigation of two-point correlations and the effect of stable thermal stability, *Journal of Fluid Mechanics*, 970, A30, <https://doi.org/10.1017/jfm.2023.631>, 2023.
- Pope, S. B.: *Turbulent Flows*, Cambridge University Press, Cambridge, UK, <https://doi.org/10.1017/CBO9780511840531>, 2000.
- Rosenberg, B. J., Hultmark, M., Vallikivi, M., Bailey, S. C. C., and Smits, A. J.: Turbulence spectra in smooth-and rough-wall pipe flow at extreme Reynolds numbers, *Journal of Fluid Mechanics*, 731, 46–63, <https://doi.org/10.1017/jfm.2013.359>, 2013.
- 1045 Shinozuka, M. and Deodatis, G.: Simulation of Stochastic Processes by Spectral Representation, *Applied Mechanics Reviews*, 44, 191–204, <https://doi.org/10.1115/1.3119501>, 1991.
- Syed, A. H. and Mann, J.: A model for low-frequency, anisotropic wind fluctuations and coherences in the marine atmosphere, *Boundary-Layer Meteorology*, 190, 1, <https://doi.org/10.1007/s10546-023-00850-w>, 2024a.
- Syed, A. H. and Mann, J.: Simulating low-frequency wind fluctuations, *Wind Energy Science*, 9, 1381–1391, <https://doi.org/10.5194/wes-9-1381-2024>, 2024b.
- 1050 Tennekes, H. and Lumley, J. L.: *A first course in turbulence*, MIT press, <https://doi.org/10.7551/mitpress/3014.001.0001>, 1972.



- Tomkins, C. D. and Adrian, R. J.: Energetic spanwise modes in the logarithmic layer of a turbulent boundary layer, *Journal of Fluid Mechanics*, 545, 141–162, <https://doi.org/10.1017/S0022112005006397>, 2005.
- Townsend, A. A. R.: *The structure of turbulent shear flow*, Cambridge university press, 1976.
- 1055 Van der Laan, M. P. and Andersen, S. J.: The turbulence scales of a wind turbine wake: A revisit of extended k-epsilon models, in: *Journal of Physics: Conference Series*, vol. 1037, p. 072001, IOP Publishing, <https://doi.org/10.1088/1742-6596/1037/7/072001>, 2018.
- Vinnichenko, N. K.: The Kinetic Energy Spectrum in the Free Atmosphere—1 Second to 5 Years, *Tellus*, 22, 158–166, <https://doi.org/10.1111/j.2153-3490.1970.tb01517.x>, 1970.
- Wint´erstein, S. R.: Nonlinear Vibration Models for Extremes and Fatigue, *Journal of Engineering Mechanics*, 114, 1772–1790, [https://doi.org/10.1061/\(ASCE\)0733-9399\(1988\)114:10\(1772\)](https://doi.org/10.1061/(ASCE)0733-9399(1988)114:10(1772)), 1988.
- 1060 Yalla, G. R., Brown, K., Cheung, L., Houck, D., deVelder, N., and Hamilton, N.: Spectral proper orthogonal decomposition of active wake mixing dynamics in a stable atmospheric boundary layer, *Wind Energy Science*, 10, 2449–2474, <https://doi.org/10.5194/wes-10-2449-2025>, 2025.
- Yamazaki, F. and Shinozuka, M.: Digital Generation of Non-Gaussian Stochastic Fields, *Journal of Engineering Mechanics*, 114, 1183–1197, [https://doi.org/10.1061/\(ASCE\)0733-9399\(1988\)114:7\(1183\)](https://doi.org/10.1061/(ASCE)0733-9399(1988)114:7(1183)), 1988.
- 1065 Zhai, C.: A novel idea: Calculating anisotropic turbulence only by Kolmogorov structure constant C_n^2 and power law α , *Results in Physics*, 19, 103 483, <https://doi.org/10.1016/j.rinp.2020.103483>, 2020.